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THE LAW OF THE SUBJECT: ALAIN BADIOU, LUITZEN BROUWER AND THE KRIPKEAN ANALYSES OF FORCING AND THE HEYTING CALCULUS

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Abstract: One of the central tasks of Badiou's *Being and Event* is to elaborate a theory of the subject in the wake of an axiomatic identification of ontology with mathematics, or, to be precise, with classical Zermelo-Fraenkel set theory. The subject, for Badiou, is essentially a free project that originates in an event, and subtracts itself from both being qua being, as well as the linguistic and epistemic apparatuses that govern the situation. The subjective project is, itself, conceived as the temporal unfolding of a 'truth'. Originating in an event and unfolding in time, the subject cannot, for Badiou, be adequately understood in strictly ontological, i.e. set-theoretical, terms, insofar as neither the event nor time have any place in classical set theory. Badiou nevertheless seeks to articulate the ontological infrastructure of the subject within set theory, and for this he fastens onto Cohen's concepts of genericity and forcing: the former gives us the set-theoretic structure of the truth to which the subject aspires, the latter gives us the immanent logic of the subjective procedure, the 'law of the subject'. Through the forcing operation, the subject is capable of deriving veridical statements from the local status of the truth that it pursues. Between these set-theoretic structures, and a doctrine of the event and temporality, Badiou envisions the subject as an irreducibly diachronic unfolding of a truth subtracted from language, a subject which expresses a logic quite distinct from that which governs the axiomatic deployment of his classical ontology. This vision of the subject is not unique to Badiou's work. We find a strikingly similar conception in the thought of L.E.J. Brouwer, the founder of intuitionist mathematics. Brouwer, too, insists on the necessary subtraction of truth from language, and on its irreducibly temporal genesis. This genesis, in turn, is entirely concentrated in the autonomous activity of the subject. Moreover, this activity, through which the field of intuitionistic mathematics is generated, expresses a logical structure that, in 1963, Saul Kripke showed to be isomorphic with the forcing relation. In the following essay, I take up an enquiry into the structure of these two theories of the subject, and seek to elucidate both their points of divergence and their strange congruencies; the former, we will see, primarily concern the position of the subject, while the latter concern its form. The paper ends with an examination of the consequences that this study implies for Badiou's resolutely classical approach to ontology, and his identification of ontology as a truth procedure.

Keywords: Badiou; Brouwer; Intuitionism; Forcing; Genericity; Subject

> 'There are two labyrinths of the human mind: one concerns the composition of the continuum, and the other the nature of freedom, and both spring from the same source—the infinite'.[1]
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> —G.W. Leibniz

One of the principal stakes of Alain Badiou's *Being and Event* is the articulation of a theory of the subject against the backdrop of the thesis that ontology, the science of being *qua* being, is none other than axiomatic set theory (specifically, the Zermelo-Fraenkel axiomatization ZF). In accordance with this thesis, every presentation of what there is—every *situation*—is held to be thought 'in its being' when thought has succeeded in formalizing that situation as a mathematical *set*.[2] The formalization of the subject, however, proceeds somewhat differently. Badiou insists that set theory alone cannot furnish a complete theory of the subject, and that for this task one needs the essentially non-mathematical concepts of time and the event. It is nevertheless possible, Badiou maintains, to determine the set-theoretical form of the subject's ontological infrastructure—the form of its 'facticity', to borrow a term from Sartre.[3] In *Being and Event,* the sought-after structures are declared to be found in the two concepts that Paul J. Cohen develops in his work on the continuum hypothesis: *genericity* and *forcing*. In a nutshell, a *generic subset* is one that cannot be discerned by any formula expressible in the model of which it is a subset. Badiou employs the notion of a generic subset to formalize the ontological structure of what he calls 'truths'. A truth, in brief, is a multiplicity that is fully immanent to the situation of which it is a truth, but which is not individuated as an element of the situation, and cannot be discerned by the linguistic and epistemic regimes proper to the situation in question. The point at which the concept of truth diverges from the concept of the generic consists in the fact that the elements of a truth will have been connected, through the work of a subject, to an *event* (*i.e.*, a rare ontological dysfunction

1. Gottfried Wilhelm Freiherr von Leibniz, 'On Freedom', in G.H.R. Parkinson (ed.), *Philosophical Writings*, trans. Mary Morris and G.H.R. Parkinson, London, J.M. Dent & Sons, Ltd., 1973, p. 107.

2. cf. Alain Badiou, *Being and Event*, trans. Oliver Feltham, London, Continuum, 2005, p. 130: 'Set theory, considered as an adequate thinking of the pure multiple, or of the presentation of presentation, *formalizes* any situation whatsoever insofar as it reflects the latter's being as such; that is, the multiple of multiples which makes up any presentation. If, within this framework, one wants to formalize *a* particular situation, then it is best to consider *a* set such that its characteristics—which, in the last resort, are expressible in the logic of the sign of belonging alone, $\in$—are comparable to that of the structured presentation—the situation—in question'. Further citations of this source will be abbreviated BE.

3. Historically, the term 'facticity' was first brought into philosophical currency through Martin Heidegger's early work. The sense in which I employ it here, however, is essentially Sartrean. The facticity of the for-itself (roughly: the subject) is the for-itself 'insofar as it *is*', which is to say, abstracted from its nihilating and transcending activity. See Jean-Paul Sartre, *Being and Nothingness*, trans. Hazel Barnes, New York, Philosophical Library, 1956, Part II, Chapter I, § II, pp. 79-84. Likewise, the being of the Badiousian subject is schematized by ontology as a finite part of a generic subset, but the subject must nevertheless be thought as transcending this ontological base; it is 'always in non-existent excess over its being', BE p. 235.

in which the immanent consistency of a situation is partially unhinged). The subject is conceived as a temporally unfolding, but always finite, *part* of such a truth—which is, in itself, always infinite (as a necessary precondition of its genericity). *Forcing* is a relation defined between certain sets belonging to the model in which the generic subset is articulated, and statements bearing upon an *extension* of this model, constructed on the basis of the generic subset in question. For Badiou, forcing expresses the 'law of the subject', the abstract form of the activity through which the subject transforms the situation on the basis of the truth in which she participates. These concepts provide Badiou with the mathematical framework of a vision of the subject as a figure initially 'subtracted' from language and from knowledge, but whose acts will come to have a transformative effect on both through its fidelity to the truth that it bears.

Badiou was not the first to conceive of the subject in this way; we find a strikingly analogous doctrine of the subject expressed in the work of Luitzen Egbertus Jan Brouwer.[4] Brouwer, too, envisions the subject as a temporal process expressed in mathematical concatenations subtracted from law and language, and whose manifestations 'within the bounds and in the forms peculiar to this life are irruptions of Truth'.[5] For Brouwer, the truth borne by the subject is none other than temporal unfolding of mathematics itself—or, rather, mathematics as it *ought* to be understood, once it is recognized in its proper essence as an autonomous subjective construction. This recognition necessitates a thorough transfiguration of existing mathematics, and results in a new discipline of mathematical thought which Brouwer calls 'intuitionism'. As Brouwer once remarked, some time into the course of this project,

> The Intuitionist intervention has had far-reaching consequences for mathematics. Indeed, if the Intuitionist insights prevail, considerable parts of the mathematical structure built so far are bound to collapse, and a new structure will have to be erected of a wholly new style and character; even the parts that remain will require thorough reconstruction.[6]

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4. I am aware of only one other discussion in print on the relations between the thoughts of Badiou and Brouwer. It consists in a brief but insightful endnote to Peter Hallward's *Badiou: A Subject to Truth*, and is worth reprinting here in full:

> Badiou's vehement opposition to intuitionism obscures the several things he has in common with Brouwer's original orientation. Like Badiou, Brouwer insists that 'there are no non-experienced truths' (Brouwer, 'Consciousness, Philosophy and Mathematics', in *Collected Works*, vol.1, p. 488). Like Badiou, Brouwer firmly 'separates mathematics from mathematical language, in particular from the phenomena of language which are described by theoretical logic' ('Historical Background', p. 509-10). Like Badiou, Brouwer conceives of genuine thought as subtraction from the petty negotiation of mundane interests. He seeks 'liberation from participation in cooperative trade and from intercourse presupposing *plurality of mind*' (p. 487, [my emphasis]). Also like Badiou, Brouwer pronounces the worldly calculation of 'security' to be unworthy of thought, and argues that any genuine philosophy works against 'cooperation' with the way of the world: 'In particular, [philosophy] should not cooperate with the state' (p. 487). Hallward, *Badiou: A Subject to Truth*, Minneapolis, University of Minnesota Press, 2003, p. 379 n.22.

5. L.E.J. Brouwer, 'Life, Art and Mysticism', in *Collected Works*, *Vol. 1*, Arend Heyting (ed.), Amsterdam, North-Holland Publishing Co., 1975, p. 7.

6. Brouwer, 'Mathematics, Science and Language', in Paolo Mancosu (ed.), *From Brouwer to Hilbert*, Oxford, Oxford University Press, 1998, p. 52.

An inevitable result of this intervention is that, in the field deployed by the intuitionist subject, much of the classical apparatus in which Badiou articulates the material foundation of his doctrine of the subject is dissolved. Neither the classical 'fidelity' of deduction nor the Cantorian doctrine of actual and extensionally determined infinities survives intact. In intuitionistic mathematics, mathematical existence becomes inseparably fused with subjective construction, and mathematical veracity becomes one with the demonstrative trajectory of the intuitionist subject.

Brouwer's vision of the nature of mathematical reality is, indeed, in stark opposition to Badiou's entire architectonic. In the last instance, however, this opposition boils down to a single point, concerning the place of the subject. Whereas Badiou axiomatically places mathematical reality *before* the subject, insofar as mathematical reality is the very form of presentation in general, the intuitionist subject *generates* this reality through the course of its temporal existence. Unlike Badiou's professedly materialist project, Brouwer conceives both mathematics and its subject along thoroughly idealist lines: rather than being a diagonal procedure imbricated in an already-existent, mathematically formalized situation, the subject is the generator of the situation in which it bears its truth.

When I say that the difference between the two theories of the subject is primarily a difference concerning the *place* of the subject, I mean this quite literally. As will be seen, both thinkers envision the *form* of the subject in strikingly similar terms. Everything hinges on the precise manner in which the subject is positioned with respect to the field of mathematical intelligibility, and on the precise orientation that this positioning gives to the closely interwoven themes of the subtraction from language and the procedural bearing of truth. We would remain in the dark, however, and possess little more than rather vague intuitions about this relation between the two theories, were the stage for their genuine encounter not presented to us by Saul Kripke's groundbreaking work in intuitionist semantics. In his 1963 paper, entitled 'Semantical Analysis of Intuitionistic Logic I', Kripke provides a model-theoretic interpretation of intuitionist logic. Among the results presented in that paper is an illustration of how Cohen's forcing-relation is isomorphic to intuitionistic entailment so long as the forcing conditions are not generic, in which case the relation behaves classically (obeying the Law of the Excluded Middle). The genericity of the sequence of forcing conditions, of course, is contingent on their forming a completed, infinite set. This does not take place at any point in the irreducibly temporal procedure by which the Badiousian subject faithfully adumbrates its truth.

The purpose of the following enquiry is to elucidate the intuitionist theory of the subject and the logical revolt that it proposes in mathematics, and to shed light on the enigmatic relations that obtain between the intuitionist and the Badiousian doctrines of the subject, particularly with respect to their logics, and the aforementioned isomorphy that Kripke discovered between them. I will begin with Brouwer and his cause.

§ 2

Near the beginning of the twentieth century, classical mathematics found itself beset with a number of antinomies, irrupting amidst efforts to provide analysis with a rigorous foundation in mathematical logic and a general theory of sets. The ensuing 'foundational crisis' became, in Badiou's eyes, an archetypical *event* for mathematics. A number of distinct interventions were taken up in response, each prescribing a careful reworking of mathematical 'fidelity', that is, of the disciplinary requirements necessary in order to preserve the integrity and consistency of mathematical truth. Among the more prominent schools of thought were that of logicism, originally headed by Frege and later championed by the young Russell and Whitehead in their *Principia Mathematica*, and the formalist school, whose greatest light was (and remains) David Hilbert. As Michael Dummett recounts, both sought to remedy the critical anomalies that had surfaced by supplying classical mathematics, as it currently existed, with a rigorous, but supplementary, foundation. The logicists would do this by producing a new logical infrastructure for mathematics, such that the latter would come to be understood as an extension of logic itself, as 'the manhood of logic' as Russell once quipped.[7] The formalists sought to supplement mathematics through a painstaking process of axiomatizing the existing mathematical disciplines and installing the resulting axiomatics within a 'metamathematical' superstructure in which their consistency would be evaluated.[8] Neither of these interventions cut as deeply into the tissue of mathematical practice as the one carried out by Brouwer.[9] As Dummett observes,

> Intuitionism took the fact that classical mathematics appeared to stand in need of justification, not as a challenge to construct such a justification, direct or indirect, but as a sign that something was amiss with classical mathematics. From an intuitionistic standpoint, mathematics, when correctly carried on, would not need any justification from without, a buttress from the side or a foundation from below: it would wear its own justification on its face.[10]

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7. 'Mathematics and logic, historically speaking, have been entirely distinct studies. Mathematics has been connected with science, logic with Greek. But both have developed in modern times: logic has become more mathematical and mathematics has become more logical. The consequence is that it has now become wholly impossible to draw a line between the two; in fact, the two are one. They differ as boy and man: logic is the youth of mathematics and mathematics is the manhood of logic'. Bertrand Russell, *Introduction to Mathematical Philosophy*, London, George Allen, 1963, p. 194.

8. Both of these programmes, incidentally, have since disbanded. 'In both cases', Dummett recounts,

> the philosophical system, considered as a unitary theory, collapsed when the respective mathematical programmes were shown to be incapable of fulfillment: in Frege's case, by Russell's discovery of the set-theoretic paradoxes; in Hilbert's, by Gödel's second incompleteness theorem. Of course, since the mathematical programmes were formulated in vague terms, such as 'logic' and 'finitistic' the fatal character of these discoveries was not inescapably apparent straight away; but in both cases it eventually became apparent, so that, much as we now owe both to Frege and Hilbert, it would now be impossible for anyone to declare himself a whole-hearted follower of either. Michael Dummett, *Elements of Intuitionism*, Oxford, Clarendon Press, 1977, p. 2.

9. For an outstanding historical account of the intuitionist intervention through the lens of the Kuhnian theory of science, see Bruce Pourciau, 'Intuitionism as a (Failed) Kuhnian Revolution in Mathematics', *Studies in the History and Philosophy of Science*, vol. 31, no.2, 2000, pp. 297-339. For a discussion on the relations between Kuhnian revolutions and Badiousian events, see Peter Hallward, *Badiou: A Subject to Truth*, pp. 210-214.

10. Dummett, *Elements of Intuitionism*, p. 2.

What is amiss in classical mathematics, Brouwer conjectured, is a clear ontological insight into the nature of mathematical truth and existence. Such insight has been systematically obscured by a logico-linguistic apparatus that has been abstracted from properly mathematical relations obtaining within certain finite systems, and, by force of habit, come to acquire the authority and prestige of a set of *a priori* laws.[11] In time, these laws had come to usurp genuine mathematical construction, and men had come to believe that mathematical truths could be arrived at, and mathematical existences disclosed, by what Mill once called an 'artful manipulation of language'.[12] Classical mathematics had mistaken the shadow for the prey; thought had subordinated itself to theoretical logic, and the mathematical study of infinite systems proceeded according to laws appropriate only for finite collections. Such is Brouwer's account of the road leading to the paradoxes.[13]

§ 3

Brouwer's intervention began with the gesture that he would retrospectively refer to as the First Act of Intuitionism. The First Act of Intuitionism, in Brouwer's words,

> *completely separates mathematics from mathematical language, in particular from the phenomena of language which are described by theoretical logic, and recognizes that intuitionist mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time, i.e. of the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the two-ity thus born is divested of all quality, there remains the empty form of the common substratum of all two-ities. It is this common substratum, this empty form, which is the basic intuition of mathematics.*[14]

The First Act seeks to wrest mathematical thought away from the reign of language, and found it in the subject, whose primordial form is given by a temporally conditioned 'two-ity'. The phenomenon of the two-ity, is understood as the primitive intuition of '*invariance in change*' or of '*unity in multitude*'[15] that manifests itself in time. This phenomenon is absolutely irreducible to more primitive terms; neither the One nor the subject itself is prior to it. Brouwer is quite explicit on this point, arguing that it would be inconceivable to posit the One as primary, in that any pretension of generating either a thinking subject or the field of numericity on the basis of a single term must begin by installing this term in a duality that exceeds it.[16] On the basis of the twoity unfolds the very be-

11. cf. Brouwer, 'Consciousness, Philosophy & Mathematics', in *Collected Works*, p. 492.

12. As quoted in Gottlob Frege, *The Foundations of Arithmetic*, trans. J.L. Austin, New York, Harper and Bros., 1960, §16, p. 22.

13. See Brouwer, 'Historical Background, Principles and Methods of Intuitionism', in *Collected Works*, p. 508-515.

14. Brouwer, 'Historical Background, Principles and Methods of Intuitionism', in *Collected Works*, p. 508. Brouwer's italics.

15. Brouwer, *Collected Works*, p. 96.

16. 'The first act of construction has *two* discrete things thought together (also according to CANTOR, Vortage auf der Naturforscherversammlung in Kassel 1903); F. Meyer (Verhandl. internat. Math. Congr. Heidelberg

ing of the subject, whose initial trajectory consists in the articulation of the infinitely proceeding sequence of natural numbers, and of the laws which govern this sequence. It is here that arithmetic has its origin, as a movement wholly interior to the trajectory of the subject.

The absolutely primordial character of arithmetic is essential for Brouwer. Arithmetic, as Brouwer understands it, is virtually consubstantial with the subject's very existence. In a few of his more speculative texts, Brouwer insists that prior to the apprehension of the two-ity in which arithmetic is grounded, the subject as such has not yet taken form.[17] Brouwer legitimates the subtraction of mathematical thought from language by appealing to the absolute *priority* of subjective mathematical construction to the installation of the subject in language. Language itself is conceived as something entirely secondary to the existence of singular subjects; in its essence, it is a vast apparatus of 'will-transmission', and where it attains an appearance of stability and rigour—as in the 'artificial' languages employed by the sciences, including mathematical logic—Brouwer sees only a crystallization of the social bond, a 'subtle form of ideology'.[18] Insofar as mathematics presents itself as a rigorous and highly structured form of thought that is prior to and eludes the ideological apparatus of language, it is, moreover, 'never without a social cause'.[19] When conducted rightly—intuitionistically—this cause is essentially subtractive, insofar as it restores a rigorous freedom to thought that 'transgresses the straightjacket of language'.[20] When mathematical practice is falsely subordinated to linguistic artifice, however, the causes it serves become bound up in the apparatuses of power, making these apparatuses all the more 'cunning'. Along these lines, Brouwer seeks to invest intuitionism with an ethical impetus.

§ 4

Any reader familiar with *Being and Event* will immediately notice several points of resonance between Brouwer's intervention and the Badiousian theory of the subject. Gathered together in a single statement of intervention we find the familiar themes of a subject subtracted from language, an irreducible and originary Two, and a 'discipline of time'. It will be possible to shed more light on these similarities once we have completed our analysis. For now, let us make note of a few key points.

First of all, the basis for the intuitionist subject's subtraction from language lies in

1904, p. 678) says *one* thing is sufficient, because the circumstance that I think of it can be added as a second thing; this is false, for exactly this *adding* (i.e. setting it while the former is retained) *presupposes the intuition of two-ity*; only afterwards this simplest mathematical system is projected on the first thing and the *ego which thinks the thing*', *Collected Works*, p. 96, n.1.

17. cf. 'Consciousness, Philosophy and Mathematics', in *Collected Works*, pp. 480-494.

18. Vladimir Tasíc, *The Mathematical Roots of Postmodern Thought*, Oxford, Oxford University Press, 2001, p. 48. Tasíc provides a well-informed and lucid account of Brouwer's theory of language in §2 of chapter 4 of this text.

19. Brouwer, 'Signific Dialogues', in *Collected Works*, p. 450.

20. Tasíc, *The Mathematical Roots of Postmodern Thought*, p. 47.

the priority of subjective mathematical constructions to any linguistic artifice. Language only comes to mathematics 'after the fact', and 'plays no other part than an efficient, but never infallible or exact, technique for memorizing mathematical constructions, and for suggesting them to others; so that mathematical language by itself can never create new mathematical systems'.[21] By contrast, Badiou conceives of the subtraction from language in entirely inverted terms, with the subject only coming into effect amidst an already existent linguistic apparatus. This subject, moreover, subtracts itself from language, at least in part, by means of language, in a process of diagonalization across the field of linguistic determination. For the Badiousian subject, the subtraction from language is grounded entirely a posteriori. For all his insight into a certain ethic that undeniably underlies Badiou's thought, Hallward's diagnosis of the Badiousian subject as a 'singularity' that 'creates the proper medium of its existence'[22] is thus somewhat inexact. This title is better reserved for the Brouwerian subject, the 'creative subject' of intuitionistic mathematics, who, as we will see, generates the medium of mathematical existence in a process reminiscent of the Pythagorean cosmogony, where the 'Indefinite Dyad', in a dialectic with the One, gives rise to the entire universe of Number. Unlike the Pythagorean doctrine, of course, the 'intuitionist cosmogony' is immaterial, subjectively generated, and possesses the crucial structural difference of declaring the Dyad prior to the One.

Badiou's divergence from Brouwer, with respect to the subtraction from language of the truth-bearing subject, is intimately bound up with one of the cardinal ambitions of Badiou's project: namely, to re-envision the concept of the subject in a manner 'homogeneous' with the forms that it has taken in our era,[23] which Badiou declares 'a second epoch of the doctrine of the Subject' (BE 3). The subject that is presented to us in this epoch, claims Badiou,

> is no longer the founding subject, centered and reflexive, whose theme runs from Descartes to Hegel and which remains legible in Marx and Freud (in fact, in Husserl and Sartre). The contemporary Subject is void, cleaved, a-substantial, and ir-reflexive. Moreover, one can only suppose its existence in the context of particular processes whose conditions are rigorous. (BE 3)

It is not difficult to place the Brouwerian conception of the subject squarely within the 'first epoch', which is essentially the era initiated by Descartes. The Brouwerian subject is essentially prior to the processes in which its active existence is affirmed, and of which it forms the reflexive centre as the guarantor of their validity and existence (*i.e.*, their constructibility). The conditions within which the Badiousian subject is traced come essentially *before* the subject. They are not given for the sake of the subject, nor is the subject conceived as their centre or guarantor. Hence there is a fundamental distinction in

21. Brouwer, 'Historical Background, Principles and Methods of Intuitionism', in *Collected Works*, p. 141.
22. Peter Hallward, 'Alain Badiou et la déliason absolue', in Charles Ramon (ed.), *Alain Badiou: Penser le multiple: Actes du Colloque de Bourdeaux 21-23 octobre 1999*, Paris, L'Harmattan, 2002, p. 296.
23. cf. BE, p. 2: '[I]t will be agreed that no conceptual apparatus is adequate unless it is homogeneous with the theoretico-practical orientations of the modern doctrine of the subject, itself internal to practical processes (clinical or political)'.

Badiou between the temporal unfolding of truth initiated by a subjective procedure and the axiomatically posited ontological context through which the truth-bearing subject proceeds. This context is conceived *classically*, as a pre-given and fully actual backdrop whose ontological structure is expressed by axiomatic set theory. In contrast to the subjective truth procedures that are explored by both thinkers, this axiomatically posited field has no need of being autonomously constructed by a subject of truth in order to be counted as existent. On this point Badiou holds fast to the strictly anti-Cartesian and anti-Kantian theory of science that he advanced in his youth, that 'one establishes oneself within science from the outset. One does not reconstitute it from scratch. One does not found it'.[24] Nothing could be further from the foundationalist thesis that 'mathematics is an essentially languageless activity of the mind having its origin in the perception of a move of time'.[25] Indeed, it is not unreasonable to suggest that it is with Brouwer that mathematics reaches its Cartesian apex. Never before has there been such a concerted effort to derive the entire edifice of mathematics from pure subjective introspection.

Certain resonances may also be traced between the respective roles played by temporality and the figure of the Two in the two doctrines of the subject. Both Badiou and Brouwer understand the subject as a temporally unfolding existence initiated in an irreducible occurrence of the Two. Here Badiou once again reverses the order of Brouwer's terms: the Two erupts into the pre-subjective fabric of consistent presentation in the form of an *event*, and gives rise to both the subject and to the temporality in which the subject articulates and produces the truth that it bears.[26] The essential divergence between the two theoreticians on the question of the twoity that initiates the subject, however, does not primarily concern the order of operations. It more concerns the place of the Two's occurrence. For Badiou, the dyadic event inaugurating the subject is not an omnipresent intuition, common to all experience. It is a rare ontological 'dysfunction' that takes place in the remotest corners of certain concrete situations. Far from being the founding intuition of mathematics, insofar as the event is an ontological dysfunction, it is 'external to the field of mathematical ontology' (BE 184), mathematical ontology being, for Badiou, classical mathematics itself (and particularly set theory). Insofar as every

24. Alain Badiou, *Le Concept de modèle*, Paris, Maspero, 1972, p. 42.

25. Brouwer, 'Historical Background, Principles and Methods of Intuitionism', in *Collected Works*, p. 510.

26. The structure of the event itself, like its extra-intuitive placement in a pre-subjective and material reality, bears little resemblance to anything encountered in the Intuitionistic field. Formally, it is conceived as a non-wellfounded multiplicity, a multiple whose elements consist of those of its site of occurrence X and of itself (Badiou outlines its form in the inscription $e_x = \{x \in X, e_x\}$). Such a multiple, Badiou observes, is manifestly non-constructible, for the self-membership that characterizes the event requires a certain 'antecedence to self' that is 'constructively impossible' (BE 304); one can construct the evental multiple only on condition that one has already done so. Heyting explicitly rules out the possibility of non-wellfounded sets (or 'species') in intuitionism, given that '[c]ircular definitions are excluded by the condition that the members of a species *S* must be definable independently of the definition of *S*; this condition is obvious from the constructive point of view. It suggests indeed an ordination of species which resembles the hierarchy of types', Arend Heyting, *Intuitionism: An Introduction*, 3rd ed., Amsterdam, North-Holland Publishing, 1971, p. 38.
My focus, here, however, is not so much the (para)ontological substructure of the event, but what Badiou alternatively calls its 'essence' or its 'position', and names the 'Two' (cf. BE, 206).

subject is initiated by an event, the event's exteriority to mathematics separates Badiou's theory of the subject from his mathematical ontology. The Badiousian subject is thus *not* primarily a mathematical structure, any more than mathematics is a subjective construction. Of course, as I have already stated, Badiou does seek to determine what is mathematically structured *in* the subject, but this cannot coincide with the Badiousian subject itself.

§ 5

Before proceeding any further in a comparison of the two doctrines of the subject, we must examine the intuitionist theory of constructive mathematics in greater detail, for it is there that the logic and the structure of the Brouwerian subject, and the intuitionist theory of truth, are deployed in full. Let us begin by examining what Brouwer means by *mathematical construction*. The simplest way of doing this is to contrast intuitionist mathematics with its classical counterpart.

Among the theorems of classical mathematics, the intuitionist recognizes as valid only those which can be made evident through explicit and finitely given algorithms. In other words an assertion can only be held to be true once one has provided 'an intuitively acceptable proof, that is, a certain kind of *mental* construction'.[27] Provability, truth, and existence are, in intuitionistic mathematics, inseparably fused together. The intuitionist decision to interpret mathematical being as, in every case, a matter of subjective construction is made in order to clear away the metaphysical trappings within which classical mathematics (as intuitionism understands it) has become ensnared. These trappings, moreover, are held to be responsible for the intuitive obscurity and the antinomies that have come to haunt the mathematical tradition. As Arend Heyting has it,

> [i]f 'to exist' does not mean 'to be constructed', it must have some metaphysical meaning. It cannot be the task of mathematics to investigate this meaning or to decide whether it is tenable or not. We have no objection against a mathematician privately admitting any metaphysical theory he likes, but Brouwer's program entails that we study mathematics as something simpler, more immediate than metaphysics. In the study of mental mathematical constructions 'to exist' must be synonymous with 'to be constructed'.[28]

What the intuitionist mission of distilling mathematical existence down to what is subjectively constructible gains for mathematics in clarity, it looses in scope, and 'full of pain, the mathematician sees the greatest part of his towering edifice dissolve in fog'.[29] The intuitionist, content to pluck out the eye that offends him, accepts the ensuing 'mutilation of mathematics' as a price that must be paid if mathematics is to remain faithful to the intuition which produced it. 'It can also be seen', reflects Heyting, 'as the excision of noxious ornaments, beautiful in form, but hollow in substance'.[30]

27.Dummett, *Elements of Intuitionism*, p. 7.
28. Heyting, *Intuitionism*, p. 2.
29. Weyl, 'On the new foundational crisis of mathematics', in Mancosu, p. 136
30. Heyting, *Intuitionism*, p. 11.

For the intuitionist, the identification of existence and subjective construction deploys a field of mathematical thought in which 'axioms become illusory'.[31] Where anything resembling axiomatization appears in intuitionist mathematics, it is only to provide a heuristically useful, but essentially secondary analytical task—as in Heyting's 1925 axiomatization of intuitionistic projective geometry (which was, incidentally, his dissertation project and was written under Brouwer's supervision). The intuitionist 'dis-axiomatization' of mathematics marks another strong divergence from Badiou's thought. While Badiou, like Brouwer, insists on liberating mathematics from the superstitious supposition that it concerns objects that are 'external' to mathematics, and identifying mathematical truth with the very movement of its thought, for the former, the axiom is precisely the point at which mathematical intuition in concentrated. Taking a position which he sees himself as sharing with both Gödel and Plato, Badiou insists that what we must understand by 'intuition' is precisely 'a decision of inventive thought with regard to the intelligibility of axioms'.[32] For Badiou, the decisional aspect of mathematical intuition 'is primary and continuously requires', and so

> it is pointless to try to reduce it to protocols of construction or externally regulated procedures. On the contrary, the constraints of construction (often and confusingly referred to as 'intuitionist' constraints, which is inappropriate given that the genuine advocate of intuition is the Platonist) should be subordinated to the freedoms of thinking decision.[33]

Badiou's identification of his position as a 'Platonism' deserves some comment here. Badiou's reading of Plato is at quite a distance from the 'platonism' that haunts the textbooks of the philosophy of mathematics (as well as a number of the intuitionists' essays—Dummett's text is a good example[34]). This textbook platonism, as Badiou sees it, simply gets Plato's thought wrong,

> because it presupposes that the 'Platonist' espouses a distinction between the internal and the external, knowing subject and known 'object'; a distinction which is utterly foreign to the genuine Platonic framework. [...] Plato's fundamental concern is a desire to declare the immanent identity, the co-belonging of the knowing mind and the known, their essential ontological commensurability.[35]

In many significant respects (which seem to have been as unclear to Badiou in his appraisal of intuitionism as they have been to Dummett in his appraisal of Platonism[36]),

31. Brouwer, 'The Effect of Intuitionism on Classical Algebra of Logic', in *Collected Works*, p. 551.

32. Badiou, 'Platonism and Mathematical Ontology', in *Theoretical Writings*, ed. and trans. Ray Brassier and Alberto Toscano, London, Continuum, 2003, p. 52.

33. Badiou, 'Platonism and Mathematical Ontology', p. 52.

34. For a clear example of the 'textbook platonism' that Badiou is opposing, cf. Dummett's synopsis of platonism on page 7 of *Elements of Intuitionism*: '[T]he platonist picture is of a realm of mathematical reality, existing objectively and independently of our knowledge, which renders our statements true or false'.

35. Dummett, *Elements of Intuitionism*, p. 49.

36. In a discussion of intuitionistic logic that I will examine later in this essay, Badiou writes that 'intuitionism is a prisoner of the empiricist and illusory representation of mathematical *objects*', BE, p. 249.

Badiou's version of Platonism mirrors the intuitionist vision of mathematics. It is nevertheless clear that something quite different is at stake in Badiou's understanding of the term 'intuition'. In the last analysis, this difference has everything to do with the fundamental unity of intuitionist mathematics, and its rootedness is the primordial intuition of the twoity. The movement of truth proper to intuitionism consists entirely in unfolding the truth of the twoity, that is, of time. Decisions which do not follow from this original intuition are not called for; when new decisions are necessitated, it is to more faithfully express this original ontological event. Cavaillès grasped this dimension of intuitionism quite clearly, when he wrote that for the intuitionist,

> mathematics is an autonomous becoming, 'more an act than a becoming', for which a definition at the origin is impossible but whose moments in their necessary interdependence betray an original essence. From the dyad to the elaborated theories, there is continuity and unpredictability.[37]

Whereas it is the nature of axioms to be, as far as is possible, separable from one another, and thus apprehensible as discrete decisions, every modification imposed upon intuitionist mathematics is prescribed by a fidelity to its 'original essence'. It is thus at least conceptually inadequate to refer to the 'Acts' of intuitionism as axioms in the classical sense. The essential unity that the intuitionist seeks to preserve in rejecting the axiomatic method does not trouble the Badiousian Platonist, however. For Badiou—who sees himself as following in Plato's footsteps on this point—mathematical thought required no greater unity than what is guaranteed for it by the logical exigency of non-contradictority. Against the intuitionist cloverleaf that lashes together being, thought and constructibility, Badiou proposes the classical (or better, Hilbertian) axiom that identifies 'being, thought and *consistency*'.[38] The axiomatic leaps and bounds that defy constructibility do not take leave of this broader sphere, which for the Platonist, Badiou writes, is governed not by an imperative of constructive coherency but by 'that of maximal extension in what can be consistently thought'.[39]

One of the most immediate and dramatic consequences of the intuitionist position is a rejection of the Cantorian concept of actually infinite multiplicity, insofar as the existence of such multiplicities can only ever be the thesis of an axiom.[40] Within intuitionist mathematics, where every existence is subjectively constructed by finite means, 'all infinity is potential infinity: there is no completed infinite'.[41] This thesis 'means, simply, that to grasp an infinite structure is to grasp the process which generates it, that to refer to such a structure is to refer to that process, and that to recognize the structure as being

37. Jean Cavaillès, 'On Logic and the Theory of Science', in Joseph J. Kockelmans and Theodore J. Kisiel (eds.), *Phenomenology and the Natural Sciences*, trans. Theodore J. Kisiel, Evanston, Northwestern University Press, 1970, p. 367.
38. Badiou, 'Platonism and Mathematical Ontology', p. 54. Emphasis added.
39. Badiou, 'Platonism and Mathematical Ontology', p. 54.
40. See Meditation 14 of BE.
41. Dummett, *Elements of Intuitionism*, p. 55.

infinite is to recognize that the process will not terminate'.[42] Such a thesis sets intuitionism at a clear distance from the set-theoretical underpinnings of Badiou's enterprise, in which the primacy of extensionality and the ubiquity of actual infinities reign supreme. It is interesting to observe, however, that the intuitionists frequently defend this thesis on the same grounds upon which Badiou defends the opposite position, namely, on the grounds that one must not denature the infinite by confusing its essence with that of the finite. For the intuitionist, the Cantorian 'destroys the whole essence of infinity, which lies in the conception of a structure which is always in growth, precisely because the process of construction is never completed;' in speaking of actual infinities, the Cantorian speaks of an infinite process 'as if it were merely a particularly long finite one'.[43] For Badiou, it is the advocates of a strictly potential conception of infinity who denature the infinite by viewing it only through the lens of finitude. Insofar as it 'determines the infinite within the Open, or as the horizonal correlate for a historicity of finitude',[44] the intuitionist disposition, by Badiou's lights, remains enslaved to the Romanticist tradition,[45] a tradition that must be overcome if thought is to unshackle itself from the 'cult of finitude'.

What is at stake in this dispute is, again, the centrality of the subject in the field of mathematical existence. So long as the (finite) subject is conceived as the central guarantor of every mathematical existence, the essence of such existences must be conceived with respect to their relation to the subject. From this perspective, the infinite *is* the outstripping of subjective construction. For the Cantorian, the infinite is deployed in its essence irrespective of the subject's position. The subject, here, does not participate in the construction of the infinite, but only its traversal.[46]

§ 6

The intuitionist identification of mathematical existence with construction, and of truth with demonstration, has consequences that penetrate through to the logical structure of mathematical reason itself. Because the intuitionist identifies the truth of a

42. Dummett, *Elements of Intuitionism*, p. 56.

43. Dummett, *Elements of Intuitionism*, p. 52.

44. Badiou, 'Philosophy and Mathematics: Infinity and the End of Romanticism', in *Theoretical Writings*, p. 25.

45. For an extended analysis of the relations between Brouwer's intuitionism and the Romantic tradition, see chapter 4 of Vladimir Tasíc's *The Mathematical Roots of Postmodern Thought*.

46. cf. Badiou, *Le Nombre et les nombres*, Paris, Éditions du Seuil, 1990, §3.17: 'Even if we can only *traverse* the numeric domain according to laws of progression, of which succession is the most common (but not the only one, far from it), why must it follow that these laws are constitutive of the being of number? It is easy to see why we have to 'pass' from one number to the next, or from a series of numbers to its limit. But it is at the very least imprudent to thereby conclude that number is defined or constituted by such passages. [*sc.* cf. NN, §3.18: "Certainly, the intuitionists adopt this impoverished perspective."] It may well be (and this is my thesis) that number itself *does not pass*, that it is immemorially deployed in a swarming coextensive to its being. […] For the domain of number is rather an ontological prescription incommensurable to any subject, and immersed in the infinity of infinities'.

statement with the construction that validates it, and the falsity of a statement with the construction that demonstrates its absurdity, he no longer has any grounds for maintaining that a given statement A is either true or false prior to the effectuation of the relevant construction. To uphold the contrary, he would have to maintain that a certain construction had been constructed *prior to its having been constructed*, which is nonsensical.

The most dramatic single effect of this orientation in thought is intuitionism's well-known rejection of the Law of the Excluded Middle (LEM). LEM states that, given a statement A, either A is true or else $\sim A$ is true, *tertium non datur*. The proposition 'A *or* $\sim A$' is thus classically valid for any A whatsoever. Within an intuitionistic context—where a statement must be proven if it is to be true—the general assertion, 'A *or* $\sim A$', 'demands a general method to solve every problem, or more explicitly, a general method which for any proposition **p** yields by specialization either a proof of **p** or a proof of ~**p**. As we do not possess such a method of construction, we have no right to assert the principle'.[47] The intuitionist rejection of LEM entails the rejection of its corollary, the principle of double negation. This principle states that $\sim\sim A$ is true if and only if A is, and legitimates a method of argument (quite common in classical mathematics) known as *apogogic* or *indirect* proof, whereby one takes to demonstrating the truth of A by demonstrating the absurdity of the absurdity of A (*i.e.* by demonstrating $\sim\sim A$). While intuitionism rejects the universal validity that classical mathematics gives to LEM and its consequences, it does, nevertheless, admit their legitimacy is certain special circumstances. Firstly, LEM holds for A whenever A is already a negative proposition (say, $A = \sim B$): either $\sim B$ or $\sim\sim B$ must be true, and $\sim\sim\sim B$ implies $\sim B$. This is due to the fact that the intuitionist accepts as self-evident the rule that B implies $\sim\sim B$ as well as the rule that if we have $A \rightarrow B$ then we also have $\sim B \rightarrow \sim A$. Of greater theoretical interest is the fact that LEM is also held to be valid in cases where one is operating in a strictly finite domain. The reason for this is that

> every construction of a bounded finite nature in a finite mathematical system can only be attempted in a finite number of ways, and each attempt can be carried through to completion, or to be continued until further progress is impossible. It follows that every assertion of possibility of a construction of a bounded finite character can be judged. So, in this exceptional case, application of the principle of the excluded third is permissible.[48]

Brouwer argues that the universality and a priority that have long been attributed to LEM are precisely due to habits acquired from reasoning within the bounds of finite situations.[49] Once mathematics turns to the investigation of the infinite—by which

47. Heyting, *Intuitionism*, p. 101.

48. Brouwer, 'Historical Background, Principles and Methods of Intuitionism', in *Collected Works*, p. 510.

49. In 'Intuitionist Set Theory', Brouwer claims that LEM and the axiom of solvability (that every problem has a solution),

> are dogmas that have their origin in the practice of first abstracting the system of classical logic from the mathematics of subsets of a definite finite set, and then attributing to this system an a priori existence independent of mathematics, and finally applying it wrongly—on the basis of its reputed a priori nature—to the mathematics of infinite sets. ('Intuitionistic Set Theory', in Mancosu, p. 27, n.4)

Brouwer always means, that which is forever incomplete—LEM immediately loses its intuitive ground.

Badiou devotes a few pages in Meditation 24 to a consideration of the intuitionist rejection of LEM and the correlate principle of double-negation. Badiou's position on this matter is (as one might expect) resolutely classical. His argument proceeds by first assuming the axioms of set theory as the common ground of the debate. Since every set-theoretical proposition is essentially reducible to a statement that is either of the form 'a set *x* exists, such that…' or of the form 'a set *x*, such that… does not exist', Badiou argues that to suppose that a certain statement is neither true nor false is to suppose that a certain, determinate multiplicity is neither existent nor non-existent. Such a position is insupportable, Badiou reasons, insofar as we are unable 'to determine, 'between' non-existence and existence, any specific intermediary property, which would provide a foundation for the gap between the negation of non-existence and existence' (BE 250). There is a subtle but significant error here. It lies in taking intuitionist logic to be determinately trivalent, that is, to be a logic with three determinate truth values. Only if this were the case could there be any grounds for rejecting intuitionist logic for want of an ontological 'foundation for the gap' between double negation and assertion. This is to mistake the very nature of the intuitionist identification of truth with demonstration and of existence with subjective construction. For the intuitionist, a negation ($\sim P$) is founded by a constructive demonstration of absurdity, and an affirmation (P) is founded by a constructive demonstration of veracity. When the intuitionist asserts $\sim\sim P$ but cannot assert P, it is because he has produced a construction demonstrating the absurdity of any construction demonstrating the absurdity of P, but has not (yet) constructively demonstrated P. What we have here is an existential foundation for $\sim\sim P$ alongside a *lack of foundation* for P. The existential correlate of the logical gap between $\sim\sim P$ and P is not an determinate intermediary between existence and non-existence, but a simple *indetermination* of existence, by which the intuitionist always means subjective construction. Badiou elides this point by situating the argument from the outset in the context of axiomatic set theory, where existence and non-existence are distributed universally and bivalently.

This elision may be permissible within the context of Badiou's enterprise, which takes as its ontological backdrop the entirety of classical set theory. It is possible to read Badiou's remarks on intuitionistic logic as an explanation of why the Law of the Excluded Middle, and consequently the deductive method of apogogic proof, is valid

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Elsewhere, in 'Consciousness, Philosophy and Mathematics', he writes that

> The long belief in the universal validity of the principle of the excluded third in mathematics is considered by intuitionism as a phenomenon of history of civilization of the same kind as the old-time belief in the rationality of π or in the rotation of the firmament on an axis passing through the earth. And intuitionism tries to explain the long persistence of this dogma by two facts: firstly the obvious non-contradictority of the principle for an arbitrary single assertion; secondly the practical validity of the whole of classical logic for an extensive group of *simple every day phenomena*. The latter fact apparently made such a strong impression that the play of thought that classical logic originally was, became a deep-rooted habit of thought which was considered not only as useful but even aprioristic. *Collected Works*, p. 492

within the (meta)ontological framework of *Being and Event*. What is less acceptable is his rather vacuous claim that in the rejection of LEM,

> intuitionism has mistaken the route in trying to apply back onto ontology criteria of connection which *come from elsewhere*, and especially from a doctrine of mentally effective operations. In particular, intuitionism is a prisoner of the empiricist and illusory representation of mathematical objects. (BE 249)

The claim that intuitionism draws its rules from the study of mentally effective operations is fair enough; indeed, on this point Badiou is in consensus with most active intuitionists (including, it seems, both Brouwer and Heyting). The claim that these rules come from 'elsewhere' than the domain of ontology, however, simply reasserts Badiou's axiomatic thesis that ontology *is* classical set theory; in this respect, the claim is a trivial one, since no one is arguing that intuitionist logic naturally emerges from classical set theory. But let us not move too quickly here—after all, the initial thesis of *Being and Event* is that '*mathematics*, throughout the entirety of its historical becoming, pronounces what is expressible of being qua being' (BE 8). Whether this mathematics is classical or intuitionistic demands a second decision; it is not decided in advance by Badiou's arguments (which are themselves more like axioms) that the presentation of being is intelligible only in terms of pure multiplicity. Intuitionist mathematics, too, proposes an ontology in which every existence is realized as multiplicity, drawing out a sort of idealist Pythagorean cosmogony without an originary One (but rather a Two). In any case, the fact that intuitionism can be said to derive its rules from mentally effective operations does not preclude the thesis that these rules correctly prescribe what is *expressible* of being; the intuitionist ontologist would have no difficulty in turning the tables here, for she is always entitled to retort that the classical ontologist applies, onto being, rules which come from a doctrine of *mentally effective operations regarding finite collections*, an accusation which is twice as damning when the rights to an ontology of infinite multiples are at stake. Badiou's suggestion, which is not taken any further than what is quoted here, that intuitionistic logic remain beholden to 'the empiricist and illusory representation of mathematical objects' is rather queer. The entire intuitionist programme takes its point of departure in seeking to overcome the 'observational standpoint' that had become the spontaneous philosophy of mathematicians, and which treated mathematical judgments as if they were judgments concerning *objects*.[50] The whole intuitionist effort is to remain faithful to a vision of mathematics as an autonomous activity of the subject, without reference to any external object. It is strange that Badiou neglects to mention this; he shares essentially the same project.[51]

§ 7

The full consequences of the intuitionist position concerning the non-predetermi-

50. Brouwer, 'Historical Background, Principles and Methods of Intuitionism', in *Collected Works*, p. 508.

51. This is the nature of Badiou's 'Platonism', as discussed in § 5 above. (cf. Badiou, 'Platonism and Mathematical Ontology', pp. 49-58.

nacy of truth can most easily be grasped by illustrating a model for this logic, such as Kripke has done in his 1963 paper, 'Semantical Analysis of Intuitionistic Logic I'. In this text, Kripke develops a model-theoretic treatment of Heyting's formalization of intuitionistic logic.[52] In all justice, before proceeding any further, we must note that both Kripke and Heyting's endeavours are, in a certain sense, external to intuitionism proper. They are formal abstractions made on the basis of intuitionist mathematics, and, according to the intuitionistic ethos, cannot be taken as expressing the essence of intuitionist mathematics itself. The remarks that Heyting makes to this effect at the beginning of his 'Formal Rules of Intuitionistic Logic' are worth repeating here. 'Intuitionistic mathematics', he writes,

> is a mental activity [*Denktätigkeit*], and for it every language, including the formalistic one, is only a tool for communication. It is in principle impossible to set up a system of formulas that would be equivalent to intuitionistic mathematics, for the possibilities of thought cannot be reduced to a finite number of rules set up in advance. [...] For the construction of mathematics it is not necessary to set up logical laws of general validity; these laws are discovered anew in each single case for the mathematical system under consideration.[53]

Motivated by the wish to 'facilitate the penetration of intuitionistic concepts and the use of these concepts in research',[54] Heyting nevertheless proceeds to abstract the general deductive structure from intuitionistic mathematics. The result is a propositional and a predicate calculus, presented in the familiar symbolic style, in which the logical consequences of the intuitionistic position are systematically unfolded. Kripke's project arose as an effort to provide a model theory for Heyting's logical calculi, and in doing this he veered even further from the orthodox path of intuitionism by constructing his model within classical mathematics. But even the embeddedness of Kripke's model in a classical framework is not the essential problem here. Kripke himself provides several indications on how the construction of the model may be conducted 'intuitionistically', and his decision to employ classical procedures is primarily a matter of expediency (it is almost always simpler to produce a classical demonstration than an intuitionistic one). It is rather that, for the intuitionist, the entire notion of a *model* is altogether secondary. But what is this notion?

Briefly put, the role of a model (in formal semantics) is to determine the veracity and soundness of a mathematical or logical system by producing a 'model structure' in which the sentences of the system can be shown to be true when they are interpreted as referring to the objects in the domain. A mathematical model thus consists of a formally specified domain of 'objects' (usually sets, subsets and relations) and a function of correspondence, called the *interpretation* of the model, established between these objects

52. Kripke provides a model for both the propositional and the predicate calculus for intuitionistic logic. In what follows, however, we will restrict our attention to the propositional calculus for the sake of simplicity and brevity.

53. Heyting, *Intuitionism*, p. 311.

54. Heyting, *Intuitionism*, p. 311.

and the syntactic elements of the system in question. As Badiou recognizes in *Le Concept de modèle*, the model-theoretic schematization of truth as ruled correspondence comes deceptively close to the empiricist or 'observational' paradigm, which makes of truth a correspondence with external objects.[55] This is precisely the orientation in thought that intuitionism seeks to overcome by identifying truth with the subjective movement of demonstrative construction, without reference to any external object. Nevertheless, Badiou, for his part, seeks to rescue the concept of the model from its empiricist appropriation, and forcefully argues that what is at issue in the mathematical employment of models is in no sense a reproduction of the 'observational' or 'empiricist' dichotomy between propositions and objects. Model theory does not concern the relation between mathematics and its exterior. Essentially, this is because both the model structure and the interpretation by which the formal system in question is evaluated are themselves produced entirely *within mathematics*. 'Semantics', accordingly,

> is an *intramathematical* relation between certain refined experimental apparatuses (formal systems) and certain 'cruder' mathematical products, which is to say, products accepted, taken to be demonstrated, without having been submitted to all the exigencies of inscription ruled by the verifying constraints of the apparatus.[56]

The use of models, in this view, is nothing other than a mode of mathematics' historical reflexivity, and is fully immanent to mathematical thought; nowhere does mathematics call upon 'external' objects to ratify mathematical knowledge.

Insofar as the use of Kripke's semantical analysis nevertheless deviates somewhat from the spirit of intuitionism, this deviation only facilitates our own enquiry. It allows us to establish a common mathematical terrain on which certain formal aspects of both Badiou's and Brouwer's theories of the subject can be drawn out. The presentation that we will give of Kripke's semantics will, necessarily, be an abbreviated one.

Like any model, Kripke's consists of two distinct components. First, we have the *model structure*, which is defined as a set **K**, a designated element $\mathbf{G} \in \mathbf{K}$, and a reflexive, transitive relation **R** defined over **K**. **G** is uniquely specified as the 'root' of the relation **R**, so that there exists no **H** in **K** such that **HRG** (**G** is '**R**-minimal' in **K**). Second, we have the *interpretation function* $\phi(P, \mathbf{H})$, where P ranges over propositions in the Heyting calculus and **H** ranges over elements of **K**. The values of this function range over the set $\{\mathbf{T}, \mathbf{F}\}$ (make no hasty assumptions here!). We also impose the condition that, given any two elements **H** and **H**' such that **HRH'**, $\phi(P,\mathbf{H}) = \mathbf{T}$ implies $\phi(P,\mathbf{H'}) = \mathbf{T}$. That is to say, the relation **R** preserves truth-values.

We will assume that ϕ has assigned a value from $\{\mathbf{T}, \mathbf{F}\}$ to each atomic proposition in the logic. In doing this, however, it is crucial to note that while the value **T** serves to represent intuitionistic truth (demonstrability), **F** *does not immediately represent intuitionistic falsity* (demonstrable absurdity). It signifies only the *absence* of a construction verifying the proposition in question (call it P), an absence which will only crystallize into the

55. See Alain Badiou, *Le Concept de modèle*, Paris, Maspero, 1972, chapters 4 & 5.
56. Badiou, *Le Concept de modèle*, p. 53.

knowledge that P is *intuitionistically false* once it has been ascertained that no **H** exists such that $\phi(P,\mathbf{H}) = \mathbf{T}$… but this comes later. The point to be made here is that the exhaustive assignment of truth-values to the atomic formulae of the logic does not contradict the intuitionist rejection of the classical vision of pre-determinate truth on which LEM rests. The formulae receiving the assignment **F** are precisely those whose truth has not yet been determined as either true or false.

The semantic values for complex sentences are defined by induction over the length of formula, in accordance with the following rules for the connectives in the logic. These are defined as follows:

a. $\phi(A \,\&\, B, \mathbf{H}) = \mathbf{T}$ iff $\phi(A, \mathbf{H}) = \phi(B, \mathbf{H}) = \mathbf{T}$; otherwise, $\phi(A \,\&\, B, \mathbf{H}) = \mathbf{F}$.
b. $\phi(A \textit{ or } B, \mathbf{H}) = \mathbf{T}$ iff $\phi(A, \mathbf{H}) = \mathrm{T}$ or $\phi(B, \mathbf{H}) = \mathbf{T}$; otherwise $\phi(A \textit{ or } B, \mathbf{H}) = \mathbf{F}$.
c. $\phi(A \rightarrow B, \mathbf{H}) = \mathbf{T}$ iff for all $\mathbf{H'} \in \mathbf{K}$ such that $\mathbf{H}R\mathbf{H'}$, $\phi(A, \mathbf{H'}) = \mathbf{F}$ or $\phi(B, \mathbf{H'}) = \mathbf{T}$; otherwise, $\phi(A \rightarrow B, \mathbf{H}) = \mathbf{F}$.
d. $\phi(\sim A, \mathbf{H}) = \mathbf{T}$ iff for all $\mathbf{H'} \in \mathbf{K}$ such that $\mathbf{HRH'}$, $\phi(A, \mathbf{H'}) = \mathbf{F}$; otherwise, $\phi(\sim A,\mathbf{H}) = \mathbf{F}$.[57]

As Kripke notes, the conditions for conjunction ('&') and disjunction ('*or*') are 'exact analogues of the corresponding conditions on classical conjunction and disjunction' (94). The conditions for implication and negation, however, significantly differ from their classical counterparts. For example, in order to assert the negation of A with respect to such and such a structure, it is necessary to ascertain that no possible extension of this structure is capable of verifying A. This particular point should be born in mind; we will encounter it again elsewhere. The condition imposed on implication serves to provide the *if… then…* relation with a certain intuitive concreteness which, as any undergraduate student in philosophy will no doubt testify, is lacking in classical logic. Intuitionistically, we may only affirm propositions of the form 'if A then B' when it is possible to constructively transform any construction verifying A into one verifying B. In Kripke's semantics, this notion is expressed by allowing $A \rightarrow B$ to be verified by a structure **H** only when any extension **H'** of **H** preserves this implication.

It is possible to illustrate these logical structures, as Kripke does, by means of a diagram. The tree-like structure in *figure 1* is an intuitionistic model for a formula A comprised of the above connectives and the atomic sub-formulae P, Q, and R.[58]

57. I quote these conditions almost verbatim from Kripke's text, altering only a few of the connective symbols to conform to the rest of this paper and the logical notation used by Badiou.
58. *Figure 1* is taken from Kripke, 'Semantical Analysis of Intuitionistic Logic I', p. 98.

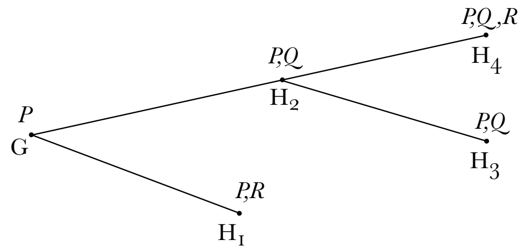

figure 1

In the model diagrammed above, we have taken **G**, $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{H}_3$, and $\mathbf{H}_4$ to be the elements of **K**. Here, they are the nodes of our tree. The relation **R** is represented by lines of succession in the tree, so that when **HRH'** we have a pathway proceeding from **H** to **H'**. Note that the **R**-minimal element **G** is at the 'root' of the tree. In the diagram, the letter of an atomic formula F is written above a node $\mathbf{H}_n$ when we have $\phi(F, \mathbf{H}_n) = \mathbf{T}$; when $\phi(F, \mathbf{H}_n) = \mathbf{F}$, F does not appear above $\mathbf{H}_n$. As Kripke has it,

> We intend the nodes **H** to represent points in time (or 'evidential situations'), at which we may have various pieces of information. If, at a particular point **H** in time, we have enough information to prove a proposition *A*, we say that $\phi(A,\mathbf{H}) = \mathbf{T}$; if we lack such information, we say that $\phi(A,\mathbf{H}) = \mathbf{F}$. If $\phi(A,\mathbf{H}) = \mathbf{T}$ we can say that *A* has been *verified* at the point **H** in time; if $\phi(A,\mathbf{H}) = \mathbf{F}$, then *A* has *not been verified* at **H**. […] If **H** is any situation, we say **HRH'** if, as far as we know, at the time **H**, we may later get information to advance to **H'**.[59]

Kripke's apparatus succeeds in capturing the temporal dimension that, intuitionism insists, must condition any logical reasoning adequate to subjectively constructed truths. Truth, which, here, is meant only to index the existence of constructive demonstrations, is not such that it is immemorially decided for every possible proposition; propositions receive truth only when the necessary constructive verification comes to pass. An interesting feature of the intuitionist notion of logical time (if we may call it that), is that while truth is always something which must be produced through the activity of a subject in time, once produced, the truth is held to be eternally valid. The language of intuitionist mathematics, as opposed to any metalanguage through which we may wish to analyse it, is therefore 'tenseless', despite the irreducible temporality of the procedures that constitute its truths. Dummett provides a helpful example on this point.[60] For the

59. Kripke, 'Semantical Analysis of Intuitionistic Logic I', p. 98. Kripke goes on to inform the reader of the point we have made above. It nevertheless bears repeating:

> Notice, then, that **T** and **F** do not denote intuitionistic truth and falsity; if $\phi(A,\mathbf{H}) = \mathbf{T}$, then *A* has been verified to be true at the time **H**; but $\phi(A,\mathbf{H}) = \mathbf{F}$ does not mean that *A* has been proved *false* at **H**. It simply is not (yet) proved at **H**, but may be established later. (p. 98)

60. The following example is a paraphrase of Dummett, *Elements of Intuitionism*, p. 337.

intuitionist, in 1882, through the work of Lindemann, the statement 'π is transcendental' became true. Prior to 1882, no such truth existed; it is nevertheless inadmissible to claim that in 1881, say, π was *not* transcendental, for to do so employs a non-constructive form of negation: no procedure existed in 1881 that was capable of demonstrating the non-transcendental nature of π, nor did any means exist of demonstrating that no procedure *could* exist that would establish that π may be transcendental. In 1881, neither the statement 'π is transcendental' nor the statement 'π is not transcendental' were true, but neither were they false. As for the statement 'it is indeterminate whether π is transcendental or not'—this is simply not a mathematical statement.[61] It is a statement of the metalanguage. By admitting as mathematical statements only those which declare the existence of a constructive procedure, intuitionism avoids encountering contradictions between tenseless propositions concerning temporally conditioned events. In this way, intuitionism produces a logic of truths that are at once eternal and created.

Dummett's example serves also to illustrate the behaviour of negation in the Kripke model, and in intuitionistic logic in general. As I have indicated, the reason why, in 1881, it was not legitimate to affirm the non-transcendental nature of π is that no procedure existed that was capable of showing that Lindemann's proof (or some other to the same effect) was not forthcoming. This state of affairs is expressed quite well by the semantic interpretation of negation in Kripke's tree-model. 'To assert ~*A* intuitionistically in the situation **H**', Kripke writes,

> we need to know at **H** not only that *A* has not been verified at **H**, but that it cannot possibly be verified at any later time, no matter how much information is gained; so we say that $\phi(\sim A,\mathbf{H}) = \mathbf{T}$ iff $\phi(A,\mathbf{H'}) = \mathbf{F}$ for every $\mathbf{H'} \in \mathbf{K}$ such that $\mathbf{HRH'}$. (99)

The intuitionist assertion of a negative proposition is thus not merely a statement of what is not actually the case (the 'case' being the current state of what has been constructed); it is a statement on what *cannot* be the case. This 'modality of the negative' is characteristic of the intuitionistic understanding of truth and its subjective essence. We will encounter it again elsewhere.

§ 8

In Meditation 27 of *Being and Event*, and again in 'La mathématique est un pensée',[62] Badiou outlines the three great 'orientations in thought', and designates them as Constructivist, Generic and Transcendent Thought, respectively. To anyone familiar

61. cf. Heyting, *Intuitionism: An Introduction*, p. 19: 'Every mathematical assertion can be expressed in the form: "I have effected the construction *A* in my mind". The mathematical negation of this assertion can be expressed as "I have effected a construction *B* in my mind, which deduces a contradiction from the supposition that the construction *A* were brought to an end", which is again of the same form. On the contrary, the factual negation of the first assertion is: "I have not effected the construction *A* in my mind"; this statement has not the form of a mathematical assertion'.

62. Alain Badiou, *Court traité d'ontologie transitoire*, Paris, Éditions du Seuil, 1998, pp. 39-54.

with this taxonomy, it is immediately tempting to place intuitionism under the rubric of 'Constructivist Thought'. On a purely mathematical register, there is much to recommend situating intuitionism within the constructivist orientation, and it is common practice in the literature to use the expressions 'constructive mathematics' and 'intuitionist mathematics' more or less interchangeably.[63] Neither the intuitionist nor the constructivist (in Badiou's sense of the term) recognize the existence of structures which cannot be constructed on the basis of a finite algorithm, and both schools of thought insist on the restriction of all quantification to domains of already-constructed entities.[64] But we need not read far into Badiou's exposition of constructivist thought to realize that this category is somewhat ill-suited to Brouwerian intuitionism. Constructivist thought, as Badiou understands it, is 'in its essence [...] a logical grammar. Or, to be exact, it ensures that language prevails as the norm for what may be acceptably recognized' as an existent multiplicity (BE 287). Nothing could be more anathematic to Brouwer's thought. As we have seen, Brouwer's founding gesture (the First Act of Intuitionism) was to announce an uncompromizing secession of genuine mathematical activity from language.[65] This is not the heart of the matter, however. There are difficulties that must be overcome before placing intuitionism within *any* of Badiou's three orientations.

All three major orientations of thought that Badiou addresses, insofar as they can be exhibited in mathematics, are demarcated according to their treatment of Cantor's continuum problem. This problem concerns the quantitative relation between the set of natural numbers ω_0 and the real number continuum, or, more generally, between a given transfinite set ω_α and the set of its subsets $\wp(\omega_\alpha)$. The question that it poses is, on the surface, quite simple: how many points are in a line? Or, equivalently, how many subsets are included in the set of natural numbers? In 1963, Cohen showed this problem to be undecidable on the basis of the axioms of set theory. The quantitative errancy of subsets over elements in any infinite set cannot be given any measure whatsoever, but nor can it be sealed over; Cantor's theorem alone tells us that there are unconditionally more subsets in any given set than there are elements. Badiou, who in the set-theoretic

63. cf. Errett Bishop, *Foundations of Constructive Analysis*, New York, McGraw-Hill, 1967.

64. For the constructivist, Badiou writes, 'if one says "there exists...", this must be understood as saying "there exists a term named in the situation"; and if one says "for all...", this must be understood as, "for all named terms of the situation"' (BE 287). In this text, a 'name' is taken to mean a finite algorithm by which the multiple in question can be constructed.

65. Moreover, the entire 'statist' ideology that Badiou seeks to connect to the constructivist orientation of thought is quite foreign to Brouwer, who only ever held the state in the greatest suspicion and hostility, and insisted on a necessary distance to be held between true thought and the state. cf. 'Consciousness, Philosophy and Mathematics', in Brouwer, *Collected Works*, p. 487: 'Of course art and philosophy continually illustrating such wisdom cannot participate in the cooperation, and should not communicate with cooperation, in particular should not communicate with the state. Supported by the state, they will lose their independence and degenerate' (*Collected Works*, p. 487). The reason why mathematics is not included in this prescription is clear enough from Brouwer's previous remarks on the matter: mathematics, by its very nature, subtracts itself from the worldly concerns of the state. By its very nature, 'the basic intuition of mathematics is left to free unfolding. This unfolding is not bound to the exterior world, and thereby to finiteness and responsibility', *Collected Works*, p. 484.

axioms sees the Platonic Ideas of ontology, interprets this mathematical impasse as a real and irreducible gap in being as such, a gap which can only be provisionally surmounted by means of a pure, subjective decision. The three great orientations in thought each propose a means of sealing this fissure, or, at the very least, a means of accounting for its origin (BE 283). The Transcendent orientation 'searches to fix a stopping point to errancy by the thought of a multiple such that it organizes everything which precedes it' (BE 283); in the context of set theory, this tendency is exhibited by the invention of axioms instituting the existence of 'large cardinals', transfinite numbers vastly outstripping anything that can be produced by means of the ordinary set-theoretic axioms. As Badiou understands it, this practice is a mathematical analogue to having recourse to the 'eye of God'. By introducing such colossal infinities into the set theoretic axiomatic, one hopes to deploy the resources that are necessary for providing an exact measure of $\wp(\omega_0)$. The Constructive or 'grammarian' orientation proposes a solution to the same problem through the aforementioned restriction of the existent to the predicatively specifiable. In set theory, this orientation is manifested in Gödel's constructible model for the axioms. This model is a hierarchical construction, which takes the empty set as its primitive stratum and generates each subsequent stratum by taking as elements all subsets of the previous stratum that can be specified by a formula restricted to that stratum.[66] The result is a standard model for set theory that validates the equation $|\wp(\omega_n)| = \omega_{n+1}$, stating that the power set of any transfinite cardinal ω_n is precisely the next largest transfinite cardinal. The third orientation, which Badiou names Generic Thought, does not so much seek to seal the gap in being so much as it seeks to disclose the 'origin' of the 'mystery of excess' (BE 283). 'The entire rational effort' of this orientation 'is to dispose of a matheme of the indiscernible, which brings forth in thought the innumerable parts that cannot be named as separate from the crowd of those which—in the myopic eyes of language—are absolutely identical to them' (BE 283). The Generic Orientation finds its mathematical expression in Cohen's work on the continuum problem, which proceeds to show that if we admit certain carefully delineated 'indiscernible' or 'generic' sets into a model for set theory, we produce new models in which the power of the continuum exceeds that of the natural numbers by 'as much as one likes', so that the power set of ω_0 may be assigned any cardinality at all that is greater than ω_0 (with the single exception that the cardinal selected not be cofinal with ω_0; that is, it cannot be $\omega\omega_0$).[67]

66. See Meditation 29 of BE for a more comprehensive treatment of Gödel's proof. Gödel's own presentation of his results can be found in volume 2 of his *Collected Works*.

67. Badiou presents Cohen's results in Meditations 33, 34 and 36 of BE. Cohen's most accessible presentation of his work is to be found in: Paul Cohen, *Set Theory and the Continuum Hypothesis,* New York, W.A. Benjamin, 1966.

Of the three Great Orientations, the Generic Orientation comes closest to Badiou's own project, and he seizes upon Cohen's concept of generic subsets in order to provide the subject and the truth that it expresses with its ontological infrastructure. Nevertheless, Badiou wishes to distance himself somewhat from the Generic Orientation as such, and sees his own work as pursuing a fourth way, one that is 'transversal to the three others', and which

holds that the *truth* of the ontological impasse cannot be seized or thought in immanence to ontology

The chief difficulty that confronts us in placing intuitionism under any of Badiou's three (or four) rubrics is that the impasse to which they respond, understood as Cantor's continuum problem, is strictly speaking, is *invisible* to intuitionist mathematics.[68] This is because the problem is premised on the hypothesis that it is legitimate to treat the continuum as a completed, transfinite set of discrete entities, be they points or subsets of natural numbers. The intuitionists hold this hypothesis to be inadmissible.[69] Their position on this matter draws its force from their insistence on the unbridgeable nature of the gaps between finitude and the infinite on the one hand, and between the discrete and the continuous on the other.[70] For the intuitionist, the classical image of the continuum as an actual and determinate, infinite set of points (or of sets of natural numbers) is cursed twice over. The question concerning the 'quantity' of such a set is therefore never raised within the intuitionist field. It would nevertheless be wrong to assume from this that intuitionism ignores the continuum altogether, or places it outside the legitimate field of mathematical thought. On the contrary, it is with respect to the older and more general problem regarding what can be said of the relation between the continuum and the natural numbers that intuitionism has produced many of its most significant innovations. And, true to the spirit of Badiou's text, it is in this field that intuitionism finds itself

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> itself, nor to speculative metaontology. It assigns the un-measure of the state [*sc.* the set of subsets, or the real number continuum when the set under consideration is ω_o] to the historial limitation of being […]. Its hypothesis consists in saying that one can only *render justice* to injustice from the angle of the event and intervention. There is thus no need to be horrified by an un-binding of being, because it is in the undecidable occurrence of a supernumerary non-being that every truth procedure originates, including that of a truth whose stakes would be that very un-binding. (BE pp. 284-5)

68. Pourciau, p. 317.

69. B. Madison Mount has produced an outstanding essay on Badiou's notion of constructivism and his application of this category to Leibniz's thought. Mount uncovers a state of affairs that is not unlike the one we find here:

> The continuum, for Leibniz, is in no way made up of points: monads, which, as Badiou notes, are sometimes equated to 'metaphysical points', are the true substratum of the spatiotemporal extensa that 'exist' only illusorily. But this does not mean, as Badiou claims, that the monad is that which can be multiplied over transfinitely to reach the continuum, subjugating the 'discontinuities' to the 'commensurable' by way of language.
>
> Instead […] the continuum persists in its incommensurability; its 'ideality' is not a simple negation of the real, but a positive quality *in actu* which prevents the adequacy of any linguistic representation […]. If it is necessary to find a successor for Leibniz in modern philosophy of mathematics, it may be less the 'constructivist orientation' than the intuitionism of Brouwer and Heyting, for whom the continuum was paradoxically best described as a *dis-continuity*, a jump beyond numeration for which no mathematical schema can fully account—'the "between," which is not exhaustible by the interposition of new units and which therefore can never be thought of as a mere collection of units'. Brouwer, 'Intuitionism and Formalism', in Paul Bernacerraf and Hillary Putnam (eds.), *Philosophy of Mathematics: Selected Readings*, 2nd ed. Cambridge, Cambridge University Press, 1983, p. 80. B. Madison Mount, 'The Cantorian Revolution: Alain Badiou on the Philosophy of Set Theory', *Polygraph*, vol. 17, 2005, p. 87.

One is left to wonder on whose foot the constructivist shoe fits.

70. On the second of these two gaps, cf. Weyl, 'On the new foundational crisis of mathematics', in Mancosu, p. 95: 'The question whether the continuum is denumerable cannot seriously arise in this theory, for, according to it, there is an unbridgeable gulf between the continuum and a set of discrete elements, a gulf that excludes any comparison'.

driven to elaborate its doctrine of the mathematical subject in such a way as to outstrip the grammarian apparatus, which otherwise, against all intentions, it weakly imitates.

§ 9

In order to have a firmer grasp on what is at stake here, let us examine a fairly simple numerical model of the linear continuum, classically understood. Consider the binary tree β, whose nodes are marked by either 0s or 1s, and whose levels are enumerated by the natural numbers (the elements of ω_o). We will call a *branch* of β any sequence of nodes running from the 'root' of the tree (< >) and proceeding infinitely. Subsets α of ω_o are then defined by the branches b_α of β, according to the following convention: $n \in \alpha$ if and only if b_α has a 0 in its n^{th} place (i.e. $b_\alpha(n) = 0$). So long as we are operating within classical mathematics, we may consider β to be an *actually infinite* structure, one which has completely traversed all of the natural numbers n. Each branch b_α thus completely defines a subset of ω_o (*i.e.*, a subset of the natural numbers). Now, each subset of ω_o can be made to correspond with a sequence of rational numbers defining a real (Cantor has shown that the rationals are denumerable, so we assume such a denumeration has taken place and correlate each subset of natural numbers with a subset of the rationals). Since there are two distinct possibilities for extension at every stage of development for each branch of the tree—namely, $b_\alpha(n) = 1$ and $b_\alpha(n) = 0$—the number of subsets in ω_o must be equal to $2 * 2 * 2 *$… *ad infinitum*, or 2^{ω_0}, a classical formulation of the power of the continuum.[71] Cantor's celebrated theorem that the power of any set S is necessary less than the power of the set of subsets of S tells us that 2^{ω_0} cannot be quantitatively equal to ω_o itself, but beyond this, classical mathematics reaches a point of profound indeterminacy. Everything hinges on what is taken to be a legitimate subset of ω_o, or, to put it another way, a legitimate pathway through β. It is here that the grammarian-constructivist orientation in set theory would impose its restriction of the existent to the linguistically constructible, admitting only subsets which can be given a predicative definition with respect to what has been constructed thus far.

The early intuitionists—those working within the field deployed by the First Act—as well as a number of 'pre-intuitionists'[72] like Borel and Poincaré, managed the real number continuum in a way that was effectively similar to the 'grammarian' approach, despite a very different theoretical motivation. While they accepted that the intuitive continuum may well be beyond the reach of mathematical intelligibility, they pragmatically circumscribed the limits of what they called the 'reduced' or the 'practical' continuum, consisting of a set of points definable by constructive means. This limitation was, in part, imposed by the fact that the intuitionists could only treat multiples (*e.g.*,

71. I borrow this construction from Mary Tiles, *The Philosophy of Set Theory: An Introduction to Cantor's Paradise*, Oxford, Basil Blackwell, 1989, pp. 66-67.

72. 'Pre-intuitionist' is a term given by Brouwer to a pre-eminently empiricist group of mathematicians including Poincaré, Borel and others, who Brouwer shared a number of sympathies, especially prior to the development of his mature intuitionist programme. See 'Historical Background, Principles and Methods of Intuitionism', in *Collected Works*, p. 509.

subsets of ω_0 defining reals) as being effectively infinite if a constructively knowable law expressed their principle of generation. The point-set that the early intuitionists accepted as constructible was even smaller than the continuum outlined by the grammarian orientation; its power did not exceed the denumerable and so it could not be identified, even provisionally, with the power set of the natural numbers.[73] It does, however, suffice for a limited but serviceable extent of mathematics.

The poverty and intuitive inadequacy of the practical continuum was nevertheless troubling to the intuitionists. The desolate horizon of the reduced continuum was, for Brouwer, an obstacle that must be overcome. His response, rightly called revolutionary,[74] was to overhaul the entire conceptual apparatus in which the problem of the continuum was posed. The new theory of the continuum exploits the two fundamental principles that had kept intuitionism at a distance from the classical set-theoretical treatments of the continuum—their refusal to treat the continuum as a point-set of *any* power, and their insistence on the irreducibly *potentiality* of the continuum's inexhaustibly infinite nature. In order to do this, Brouwer recognizes, it is necessary to surpass the conceptual disposition proposed by the First Act. A Second Act of Intuition is therefore declared. This Act

> *recognizes the possibility of generating new mathematical entities:*
>
> *firstly in the form of infinitely proceeding sequences p_1, p_2, …, whose terms are chosen more or less freely from mathematical entities previously acquired; in such a way that the freedom of choice existing perhaps for the first element p_1 may be subjected to a lasting restriction at some following p_v, and again and again to sharper lasting restrictions or even abolition at further subsequent p_v's, while all these restricting interventions, as well as the choices of the p_v's themselves, at any stage may be made to depend on possible future mathematical experiences of the creating subject;*
>
> *secondly, in the form of mathematical species, i.e. properties supposable for mathematical entities previously acquired, and satisfying the condition that, if they hold for a certain mathematical entity, they also hold for all mathematical entities that have been defined to be equal to it…*[75]

The Second Act dramatically increases the power of intuitionist mathematics, and provides the groundwork for what Brouwer calls 'Intuitionist Set Theory', a discipline which, like its Cantorian counterpart, sets itself the task of charting a course through the labyrinth of the continuum. Both the path and the gauge of the Brouwerian trajectory are entirely different than those chosen by Cantor, however. If the principle challenge that Cantor selected for his theory of sets was that of providing an exact quantitative measure of the linear point set with respect to the natural numbers, the task proper

73. Mark van Atten, 'Brouwer, as Never Read by Husserl', *Synthese*, vol. 137, no. 1-2, pp. 3-19, p. 3.
74. Hermann Weyl, a German philosopher and mathematician who was once one of Brouwer's more significant allies, once famously exclaimed: 'Brouwer—that is the revolution!', Vladimir Tasíc, *The Mathematical Roots of Postmodern Thought*, p. 54. In his article on intuitionism and phenomenology, Mark van Atten writes, in a similar vein: 'Around 1917, two revolutions took place, one fake, and one true. The true one happened in mathematics, and consisted in the introduction of choice sequences by Brouwer', in 'Brouwer, as Never Read by Husserl', p. 2.
75. Brouwer, 'Historical Background…' in *Collected Works*, p. 511.

to intuitionist set theory is that of mathematically thinking the continuum in its very *indeterminacy* and errancy *vis-à-vis* discrete numeration, and to do this without letting the continuum dissolve into an unintelligible mystery. The errancy of the continuum, dispelled by the grammarian orientation, becomes a locus of mathematical investigation in intuitionism, and finds expression in the irreducibly unfinished and unforeseeable progression of free choice sequences.

In order to show how this is possible, it is necessary to specify a few of the concepts that the Second Act bequeaths us. The two new structures which the Act explicitly puts forth are *choice sequences* and *species*. A species is essentially a class, and, conceptually bears little difference from the classical notion, save for what is at issue are classes of intuitionistically admissible structures. Of choice sequences, there are two essential types. *Lawlike sequences* are infinitely proceeding sequences of natural numbers—or any other constructible mathematical structure—prescribed by a determinate algorithm or 'law'. The notion is close to what Badiou calls *discernible* sets—but we will come to this later. *Free choice sequences* are infinitely proceeding sequences that are not determined by any law or algorithm. Between the two, any variety of intermediate forms are possible, and laws may be imposed upon and removed from free choice sequences at any stage in their development as the subject so chooses. A *spread* is a species of choice sequences possessing a common 'root' or first term, and which is governed by two laws (which we will often collapse into one for the sake of brevity): first, there is the *spread law*, noted Λ_X where X is the spread in question. This law determines the admissibility of *finite initial segments* of choice sequences into the spread. Every spread law must: (1) admit the empty sequence < > as the root of the spread, (2) not admit any choice sequence possessing an inadmissible initial segment, and (3) for each admissible initial segment, admit at least one possible extension of this segment into the spread, so that every admitted segment may proceed indefinitely along at least one path. Further restrictions may be imposed to produce spreads of the desired form. The second law is called the *complementary law* of the spread, as is designated Γ_X. This law permits us to produce spreads of mathematical entities other than the natural numbers by assigning, to every admitted sequence of the spread, some other intuitionistically constructed structure. The only restriction on this law is that it be effectively decidable for every assignment.

We are now in a position to produce the intuitionistic construction of the continuum. Whereas the classical continuum is conceived as a determinate set of real numbers, the intuitionist continuum is composed of *real number generators*. There are many possible forms of these; here we will consider infinitely proceeding sequences of rational numbers $\{r_n\}$ such that $|r_n - r_{n+1}| < 2^{-n}$. Real number generators, in intuitionist mathematics, are analogous to the classical definitions of real numbers. The crucial difference lies in the fact that they are not conceived as completed infinite sets, but as intensionally determined, infinitely proceeding sequences. The intuitionist continuum is now constructed as follows: we begin by assuming an enumeration of the rational numbers $r_1, r_2, \ldots$; we then define a spread of natural numbers by the spread law Λ_C: 'Every natural number forms an admissible one-member sequence; if $a_1, \ldots, a_n$ is an admissible sequence, then

$a_1, \ldots, a_n, a_{n+1}$ is an admissible sequence if and only if $|ra_n - ra_{n+1}| < 2^{-n}$’.[76] The *complementary law* for the spread *C*, noted Γ_C, assigns the rational number ra_n to every admissible sequence $a_1, \ldots, a_n$. The spread *C* thus comprises of every possible real number generator that may be given by lawlike algorithms. Beyond these lawlike generators, however, there exists an innumerable plurality of unspecified and underdetermined choice sequences which do not yet determine, but which never cease to approach, real numbers. It is thus that ‘we have here the creation of the “continuum,” which, although containing individual real numbers, does not dissolve into a set of real numbers as finished beings; we rather have a *medium of free Becoming*’.[77] Let us note that this is a medium created by the very subject who traverses it, a subject properly called ‘singular’.[78]

It is in the Second Act that we can situate a clear break between grammarian-constructivism and intuitionism, within the mathematical framework of the latter. A free choice sequence is an intuitionistically constructible entity that is *not* constructible in the grammarian sense. By definition, a free choice sequence is determined by any constructible algorithm or predicative formula. With respect to the current enquiry, these structures are significant for the fact that, even if the subject wholly pervades intuitionistic mathematics, at no point is it more exposed than in Brouwer’s theory of free choice sequences. Everywhere else, the idealist mandate of constructibility has all the same effects as a fairly weakened strain of grammarian thought. Here alone do we have a constructive form that can be generated *only* by way of subjective decisions. It is the point where intuitionism bares its subjective essence beneath its accidentally grammarian attire.

Once outside of the scope of grammarian thought, the concept of free choice sequences would seem to direct us, instead, towards the Generic Orientation, insofar as it is the business of free choice sequences to trace out a ‘random conglomerate’ in a spread, unlegislated by any lawlike parameters beyond those that deploy the spread itself. To be more precise, we may say that the direction in which these sequences lead us more closely approaches Badiou’s subject-theoretic employment of the generic than the thought of the generic itself. In the theory of the subject presented in *Being and Event*, what we find is an attempt to mediate the mathematical concept of genericity via a concrete, subjective procedure that is infinitely proceeding in time, but which is at any moment finite. It is, in fact, not the generic at all that provides the most accurate mathematical schema for the structure of a subjective truth procedure *qua* procedure. For this we must look elsewhere.

§ 10

A subject’s existence, as Badiou has it, is always temporal, and, beginning with an act of intervention that forms an indecomposable dyad with an event, consists in travers-

76. Heyting, *Intuitionism*, p. 36.
77. Hermann Weyl, ‘On the new foundational crisis in mathematics’, in Mancosu, p. 94.
78. cf. Hallward, ‘Alain Badiou et la déliason absolue’, p. 296.

ing an infinitely complex situation through an inexhaustible process that Badiou calls a *fidelity*. The business of a fidelity consists in performing a series of *enquiries* regarding the possible 'connections' that may or may not obtain between such and such an element of the situation (schematized as a set) and the event to which the subject seeks to remain faithful. A fidelity is said to be a *truth procedure* if the projected infinite subset of the situation consisting of all the elements positively connected to the event will have been *generic*. Briefly put, a generic subset is one which cannot be separated or discerned by any formula restricted to the situation—or, more precisely, restricted to the model structure *S* in which the situation's ontological form is expressed. This means that within the situation, there exists no law that would be a necessary and sufficient condition for belonging to the truth. Now, given that the procedure always occurs *in time*, 'at every moment, an evental fidelity can be grasped in a provisional result which is composed of effective enquiries in which it is inscribed whether or not multiples are connected to the event' (BE 234), and this provisional result is always *finite*. Nothing of the genericity of the fidelity (its ontological 'truthfulness') can thereby be grasped in any such result, for so long as a set is finite it is always possible to compose a restricted formula that would be a necessary and sufficient condition for membership in that set, even if this formula is as rudimentary as $[\alpha_1 \in \mathbf{a} \; \& \; \ldots \; \& \; \alpha_n \in \mathbf{a}]$ where *n* is the number of elements in **a**. Only an infinite set has any chance of being generic, given that a formula can only be of finite length (of course, not *all* infinite sets are generic, by any stretch). If we wish to capture the mathematical essence of the Badiousian truth-procedure *in the act*, then it is clear that a strictly extensional apprehension of the subject's fidelity is insufficient, and 'in truth […] quite useless' (BE 235). In order to adequately think the essence of a fidelity, we must attend to its temporality, and thus to its 'non-existent excess over its being' and the 'infinity of a virtual presentation' towards which it projects itself.

The essentially intensional and temporal notion of choice sequences, as developed by Brouwer and his school, is of far greater worth to us here than anything offered by the atemporal and extensionally determined landscape of classical set theory. So long as we remain within time—as every subject must—it is possible to capture the ever-incomplete unfolding of the generic procedure in terms of an Brouwerian choice sequence. I now propose to do just this.

Let us begin by circumscribing the domain in which the subject will operate. According to Badiou's exposition, this consists of a set of *conditions*, noted ©, that is both an element and a subset of the fundamental situation *S* inhabited by the subject. For the sake of simplicity, let us follow Badiou's initial example and take © to consist of the empty set Ø, and sets of countable, but possibly infinite, ordered sequences of 1s and 0s (although © may be of far greater complexity in some cases). These sets are called the 'conditions' in ©. Following Badiou's notation, we will indicate such sets by the letter π, differentiating them with numerical subscripts when necessary. The generic truth that expresses the completed subjective procedure is a form of a more general type subset defined over ©, called a 'correct subset'. The configuration of these subsets is governed by two rules, noted Rd_1 and Rd_2. Rd_1 requires that if a condition π_1 belongs to a correct subset δ, then any condition π_2 that is a subset of π_1 (that is 'dominated' by π_1, as Badiou puts it) is also an element of δ. Hence, if we have $\{\langle 1\rangle, \langle 1,1\rangle\} \in \delta$, then we must also have $\{\langle 1\rangle\} \in \delta$ and $\emptyset \in \delta$. Rd_2 requires that the elements of correct sets satisfy a relation a *compatibility* amongst one another. Two conditions π_1 and π_2 are said to be *compatible* if and only if either π_1 is a subset of π_2 or π_2 is a subset of π_1. For example, $\{\langle 1\rangle, \langle 1,0\rangle\}$ is compatible with $\{\langle 1\rangle, \langle 1,0\rangle, \langle 1,0,1\rangle\}$ and with $\{\langle 1\rangle, \langle 1,0\rangle, \langle 1,0,0\rangle\}$, but not with $\{\langle 1\rangle, \langle 1,1\rangle, \langle 1,1,1\rangle\}$. In order to ensure that all of the elements of a correct subset are compatible with one another, Rd_2 requires that for every conditions belonging two δ there exists a third, also belonging to δ, of which the first two are both subsets. Formally, these two rules are written:

$$Rd_1\text{: } [\pi_1 \in \delta \;\&\; \pi_2 \in \pi_1] \rightarrow \pi_2 \in \delta$$

$$Rd_2\text{: } [(\pi_1 \in \delta) \;\&\; (\pi_2 \in \delta)] \rightarrow (\exists \pi_3)[(\pi_3 \in \delta) \;\&\; (\pi_1 \subset \pi_3) \;\&\; (\pi_2 \subset \pi_3)]$$

So far we have not yet parted ways with Badiou's own mode of exposition.[79] We will do this now, by defining a *spread* of correct subsets over ©, which we will call Δ.

In order to capture the incremental development of the correct subsets, among which those capable of characterizing truth procedures will figure, let us introduce some additional notation to Badiou's apparatus. We will write $\delta(n)$ to indicate a correct subset with n elements. This 'n' will also designate the distance of the sequence in question from the root of the spread. $\delta(m)$ will be considered an initial sequence of $\delta(n)$ when $\delta(m) \subset \delta(n)$ and $m < n$. If two sequences are not compatible (if one is not an initial sequence of the other), we will differentiate the two by subscripts (*e.g.* $\delta_1(n)$ and $\delta_2(m)$). As stated above, a spread is given to us by its *spread law* and its *complementary law*. Here, for the sake of concision, we will conflate the two, skipping the construction of a natural number spread and proceeding directly with the formation of a spread over ©; our conflated law will be denoted $\Lambda\Gamma_\Delta$. What we wish to do here is to define a spread whose sequences will all be correct parts of ©. Its law must, therefore, imply the two 'rules of correctness', Rd_1 and Rd_2. This law takes the form of a function, whose domain is the set of subsets of © (*i.e.* $\wp$(©)) and whose range is the set $\{1,0\}$. This function is constructed to return a 0 when its argument is admissible, and a 1 when it is not. The law $\Lambda\Gamma_\Delta$ is formulated as follows:

$$\Lambda\Gamma_\Delta(\delta(n)) = 0 \quad \textit{iff} \quad [\delta(n) = \{\emptyset\}] \;\; \textbf{or}$$

$$[\Delta(\delta(n\text{-}1) = 0 \;\&\; \delta(n\text{-}1) \subset \delta(n) \;\&\; (\forall \pi_1 \in \delta(n\text{-}1))(\exists \pi_2 \in \delta(n))(\pi_1 \subset \pi_2)]$$

$$\Lambda\Gamma_\Delta(\delta(n)) = 1 \quad \textit{otherwise}$$

It is a fairly simple exercise to ascertain that any sequence $\delta(n)$ admitted by this law obeys the two rules of correctness stated above. A small portion of the resulting spread is diagrammed in *figure* 2.

79. The entire theory of correct subsets is to be found in section 3 of Meditation 33 of BE.

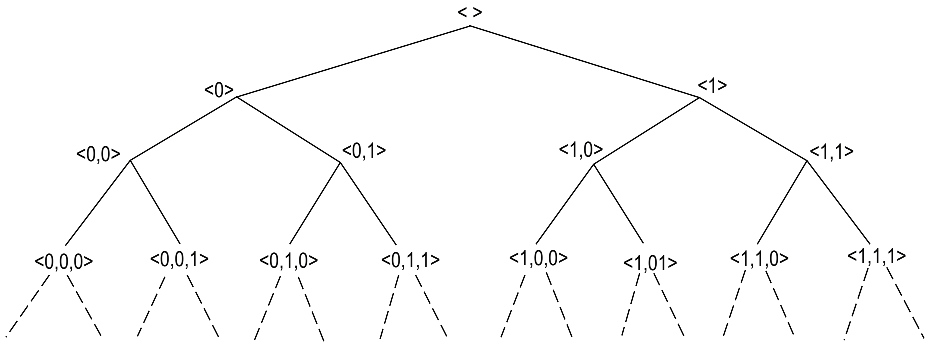

figure 2

The next step in our construction consists in delineating a *potentially generic* sequence in Δ. It must be understood that the genericity of the sequence must always remain *potential*, so long as we are operating within an intuitionist spread, for only an actually infinite sequence can be truly generic. This state of affairs is no different than that which we find in any concrete truth procedure, according to the argument advanced in *Being and Event*. The infinite multiplicity proper to a concrete exercise of fidelity—a truth-procedure—is always only 'virtual' (BE 236) or potential. That this infinity has a fully actual locus of being in the situation itself, as Badiou understands it, does not change the fact that the truth-procedure itself is internally characterized by a *potentially* infinite progression, no less than any intuitionistically admissible sequence. Even so, a significant conceptual difference between an intuitionistic sequence and a Badiousian truth-procedure is legible here; namely, that the medium of the Badiousian subject is not its own creation.

It is clear that no *lawlike* sequence is fit to express the concept of a potentially generic procedure, since a lawlike sequence is precisely one whose elements are extracted from the spread according to a constructible principle, that is, by a formula restricted to the (pre-constructed) universe in which the spread unfolds. Lawlike sequences are, in Badiou's language, essentially *discernible* sequences. One may therefore suspect that all that needs to be done to schematize a potentially generic procedure in Δ is to define that procedure as a *free choice sequence*, a sequence whose successive choices are entirely unrestricted, so long as they remain within the boundaries set by the spread law. This too, however, is insufficient, for nothing guarantees that such an arbitrary sequence will not inadvertently (*i.e.* extensionally) fall under a 'lawlike' determinant, become discernible to the situation, and fall short of genericity. Neither the 'anarchic' nor the 'legalistic' modes of operation will be sufficient for our task. The anarchic approach does, however, come somewhat closer to what we are after here. The entire problem lies in placing the necessary restrictions on the freedom of the sequence, in 'disciplining' the sequence in a way that does not rob it of its freedom, but which keeps it at a distance from the Law. Much of Badiou's own approach to the question of liberty can be gleaned from this problematic. Hallward is quite correct in observing that 'Badiou sees freedom as an exceptionally fragile achievement', quite unlike the those who, from Kant through to Sartre and Brouwer, see it 'as a necessary presumption'.[80] It is not a question, here, of empirical freedom, the condition of not being in bondage; Badiou's thesis is that rarity and fragility are characteristic of the ontological and trans-ontological basis of freedom itself: rarity, since the prerequisite unbinding from being in itself takes place only in exceptional events, and fragility, because the freedom of the subject can only sustained so long as the subject maintains the protracted effort of subtracting itself from the law. Moreover, these two conditions support one another in their being, for 'the event is only possible if special procedures conserve the evental nature of its consequences' (BE 211). Only through the genericity of the truth procedure may an event succeed in making its mark on being. No such fragility confronts the Brouwerian subject, for even the lawlike sequences are conceived in terms of choice sequences constrained by 'self-imposed restrictions'.

In order to faithfully distil the bare subjective essence of the free choice sequence from the pseudo-grammarian dross that surrounds it, and exhibit a structure that expresses the fragile and disciplined freedom that characterizes the subjective truth procedure, we must place certain restrictions on an otherwise free choice sequence. These must be sufficient to 'discipline' the sequence in a such a way that it does not allow itself to be (permanently) captured by any existing lawlike sequence, without consigning the subject to a newly invented lawlike sequence of its own. The rule that we will impose will be the following: for any lawlike sequence λ, if $\lambda(n) = ♀(n)$ then there must exist some m such that $\lambda(m) \neq ♀(m)$. Given that ♀ is denumerable (even when conceived as an actually infinite subset) and can always be effectively enumerated on the basis of the natural order germane to all correct subsets, it follows that wherever ♀ differs from a discernible correct subset, the point at which it differs can be indexed by a finite ordinal. The index m of this point, moreover, will always be constructible, since the means for its determination are already constructively given to us in the comparison of an algorithmically generated lawlike sequence and a subjectively constructed choice sequence.

An unsettling practical consequence of this prescription, which sufficiently captures the potential genericity of any concrete procedure, is that a potentially generic procedure can, consistently, remain lawlike *indefinitely*: it is always possible to procrastinate its divergence from any given lawlike sequence. It has, so to speak, 'all the time in the world' to become generic. It is therefore impossible to decide, based on empirical evidence, whether any procedure is or is not generic. Strictly speaking, the *truthfulness* of a procedure does not disclose itself in extensionally determinate evidence; it can be testified to only in the interiority of the sequence, with respect to its projected intension. Any declaration concerning the existence of a truth must, therefore, always remain hypothetical and anticipatory, without the hope of sufficient evidence ever arriving. For as long as a procedure is conceived as a stepwise concatenation of discrete elements of a situation, it is clear that never will this procedure achieve historical completion. The

80. Peter Hallward, *Badiou*, p. 167.

condition of genericity, like the holiness to which the Kantian subject aspires, is 'a perfection of which no rational being of the sensible world is capable of at any moment in his existence. Since, nevertheless, it is required as practically necessary', if the procedure is to be affirmed as a *truth*, 'it can only be found in a *progress in infinitum* towards that perfect accordance', or rather that pure *discordance*, with the Law.[81] No less than Kant, Badiou is forced to postulate a form of 'immortality' for the subject. Badiou does not balk at this exigency, and insists that in its essence, 'subjectivation *is* immortal'.[82] The immortality avowed here, however, is not that of the 'human animal' who bears the truth in question, but the *progress in infinitum* of which the subjective procedure itself is, in principle, capable, and which the truth that it serves demands of it. This illuminates a significant point concerning the Badiousian subject that we have not yet mentioned: the subject is not identical with the individual as such, but with the procedure in which the individual is engaged. There may therefore be collective subjects, just as there may be epochal subjects, whose scope far exceeds that of any single participant. All this is quite different from Brouwer's occasionally quite solipsistic tendencies. Nor does Brouwer's theory of the subject place any wager on the existence of an actual infinite, but this is quite in accordance with his radically immanentist vision of the subject and its mathematical task.

§ 11

The anticipatory nature of genericity does not prevent the subject from drawing conclusions regarding the postulated 'new world' that would come at the end of the truth procedure. This is where the operation of *forcing* comes into play. Before it is possible, however, it is necessary to calibrate the initial situation by defining within it a complex apparatus of 'names' for the elements of the new world, the 'generic extension' *S*(♀) of the initial situation *S*. These names are defined, prior to the exact determination of their referents, as sets in the initial situation of a certain kind—namely, as ordered pairs consisting of conditions in © and other, previously constructed names. In the interest of brevity, I will forgo a detailed account of how this may be done; one such method is illustrated in Meditation 34 of *Being and Event*; another is given in Chapter IV, § 3 of Cohen's *Set Theory and the Continuum Hypothesis*, and still others are available in the existing literature on the topic. As Cohen notes, the precise method chosen for the calibration of names 'is of no importance as long as we have not neglected any set' in the generic extension (Cohen, 113). In empirical situations, moreover, it is certainly to be expected that the method should differ from one specific truth procedure to another. In any case, what is essential is that the referential value of these names is determined by the composition of ♀; more precisely, the referential value of each name is determined by the member-

81. See Immanuel Kant, *Critique of Practical Reason*, trans. T.K. Abbott, Amherst, Prometheus Books, 1996, p. 148.
82. Badiou, *Ethics: An Essay on the Understanding of Evil*, trans. Peter Hallward, London, Verso, 2001, p. 12. Emphasis mine.

ship in ♀ of the conditions which enter into the composition of the name in question.

The constellation of names is generated by the subject figure in what Badiou calls the 'subject-language', an amalgam of the native language of the situation and the names whose reference is contingent on the composition of the generic truth ♀. This language is naturally empty or nonsensical for inhabitants of the initial situation *S*, since the names it employs, in general, do not have a referent in *S*; the situation to which they refer, moreover, has not yet fully arrived, and even here their referential function is filtered through what, for those in *S*, is entirely indiscernible.[83] Operating within this new language, the subject is capable of making certain hypotheses of the form: '*If* I suppose that the indiscernible truth contains or presents such or such a term submitted to the enquiry by chance, *then* such a statement of the subject-language [*sc.* bearing on the new situation, the generic extension *S*(♀)] will have had such a meaning and will (or won't) have been veridical' (BE 400). The hypothetical character of these statements is gradually, but never completely, resolved throughout the course of the generic procedure, as the elements of the truth become known to the subject in question (*i.e.*, as the index *n* of ♀(*n*) increases). Of the projected composition of ♀, 'the subject solely controls—because it is such—the finite fragment made up of the present state of the enquiries. All the rest', we are told, 'is a matter of confidence, or of knowing belief' (BE 400).

The rational means by which the subject of the generic procedure makes such assertions and hypotheses is governed by the *forcing relation*, which Badiou names as the 'fundamental law of the subject'. The ontological form of this relation derives from Cohen's work on the continuum problem, where forcing is used to demonstrate the existence of models for set theory in which the power of the continuum may exceed ω_1 by virtually any degree at all (the only restriction being $\omega_0 < |\wp(\omega_0)| \neq \omega\omega_0$). In the context of Badiou's theory of the subject,

> [t]hat a term of the situation *forces* a statement of the subject language means that the veracity of this statement in the situation to come is equivalent to the belonging of this term to the indiscernible part which results from the generic procedure. It thus means that this term, bound to the statement by the relation of forcing,

83. As Badiou describes it, this state of affairs finds a peculiar resonance in Brouwer's work. As Jan von Plato observes, Brouwer's 1920 papers on intuitionistic mathematics are populated with strange and often esoteric terminology and notation. This unusual reconfiguration of mathematical language, von Plato informs us, has its theoretical motives in the programme of the 'Signific Circle', a philosophical group in which Brouwer participated. 'The circle', he writes, 'aimed at moral betterment of humankind through a socio-linguistic reform. Brouwer himself believed that old words contain moral connotations that can lead to evil thoughts. For him, language was in the first place a means for getting power over others. Thus the strange and specifically intuitionistic vocabulary (and notation, in part still followed by some intuitionists) is part of a utopian program of language revision'. (Jan von Plato, 'Review of Dirk van Dalen, *Mystic, Geometer, and Intuitionist, The Life of L.E.J. Brouwer vol. 1. The Dawning Revolution*', in *Bulletin of Symbolic Logic*, vol.7, no.1 March, 2001, p. 64) Indeed, Brouwer considered his intuitionist movement to be, in a subtle but significant way, of both spiritual and political importance, and part of his task of 'creating a new vocabulary which admits also the spiritual tendencies in human life to considerate interchange of views and hence social organization' ('Signific Dialogues', *Collected Works*, p. 448).

belongs to the truth. (BE 403)[84]

The forcing relation is intimately related to the logical notion of *implication* or *entailment*, as Cohen points out (Cohen, 111). It determines the states of affairs that will arise on the condition that this or that set π belongs to the generic ♀ on the basis of which *S*(♀) is constructed. As Badiou has it, what is at stake here is the immanent logic of a subjective truth procedure. It is, in several respects, analogous to the logic of being—that is, the classical logical calculus by which set theory operates. Where forcing formally diverges from classical logic, it does so insofar as it is compelled to derive its veracities from an infinite sequence whose total composition is inaccessible to any algorithmic determination. It is no accident that these are precisely the exigencies faced by the intuitionist subject, when operating in a domain that cannot be finitely specified.

A definition for the forcing relation with respect to atomic formulae cannot be adequately presented within the limits of this paper. The curious reader may find a thorough treatment in Cohen's text, and an adequate gloss of the forcing of atomic formulae in Appendix 7 of *Being and Event*. What is more significant for us, in any case, is the logical structure which the forcing relation takes with respect to compound formulae. Here, the structural divergence of forcing from classical entailment is clearly legible. With respect to the propositional connectives,[85] the definition of forcing is as follows:

a. π forces *P* & *Q* if π forces *P* and π forces *Q*.
b. π forces *P or Q* if π forces *P* or π forces *Q*.
c. π forces *P* → *Q* if either π forces *P* or π forces ~*Q*.
d. π forces ~*P* if for all π' dominating π, π' does not force *P*.[86]

As Cohen remarks, these 'definitions do not imply that for π and *P* we must have either π forces *P* or π forces ~*P*. Also, forcing does not obey some simple rules of the propositional calculus. Thus, π may force ~ ~*P* and yet not force *P*'.[87] To be more precise, the definitions we have here do not obey some simple rules of the *classical* propositional calculus; as an analogue of entailment, the forcing relation here defined is, in fact, highly suggestive of the *intuitionistic* calculus. Consider the definition for negation. As Cohen tells us, it is only possible for a condition π to force ~*P* so long as no other condition

84. In more technical terms:
- if a condition π forces a statement on the names, then, for any generic part ♀ such that π ∈ ♀, the same statement, this time bearing on the referential value of the names, is veridical in the generic extension *S*(♀);
- reciprocally, if a statement is veridical in a generic extension *S*(♀), there exists a condition π such that π ∈ ♀ and π forces the statement applied to the names whose values appear in the veridical statement in question. (BE p. 412)

85. As in the above exposition of Kripke's intuitionistic semantics, I leave out the conditions for the quantifiers ∃ and ∀. Again, this is done in the interest of brevity. The interested reader may consult Chapter IV of Cohen's text.

86. These definitions are presented in Cohen, p. 117-8. I have altered some of the notation to conform to Badiou's. This, of course, does not affect the meanings of formulae in question.

87. Cohen, p. 118. Notation altered; see previous footnote.

participating in the generic sequence forces *P*. 'In forcing', Badiou observes, 'the concept of negation has something modal about it: it is possible to deny once one is not constrained to affirm'; the certainty of non-constraint always being deferred until the sequence is completed. 'This modality of the negative', Badiou continues, 'is characteristic of subjective or post-evental negation' (BE 415)—just as it is characteristic of the temporally deployed constructions of the intuitionist subject (cf. § 7). It is not merely a superficial structural similarity that is at issue here; the formal congruence between the two subjective logics is the effect of essentially identical requirements. These requirements stem from the fact that 'both' subjects participate in the articulation of a truth which finds its full determination only in time. We have seen that Badiou's temporalization of the subjective truth procedure has the effect of translating the generic subset in which the subject participates into the intuitionistically legible form of a *potentially generic choice sequence*; we see now that this same temporalization seems to bring the logic of the post-evental subject into conformity with the logic of intuitionism.

In the same 1963 paper from which we earlier drew the semantical analysis of intuitionistic logic, Kripke confirms our suspicions (pp. 118-120). He shows that model structures of the sort presented in § 7 can be, rather straightforwardly, interpreted as modelling the forcing relation instead of the Heyting calculus for intuitionistic logic. Roughly speaking, this involves assigning *forcing conditions* to the nodes of the model structure in such a way that $\pi \mathbf{R} \pi'$ whenever we have $\pi \subset \pi'$. For those π which are elements of ♀, we have $\phi(P,\pi) = \mathbf{T}$ or $\phi(P,\pi) = \mathbf{F}$, according to whether π forces *P* or fails to do so. In this model, ♀ thus appears as an infinitely long path through the tree (in the classical context of Kripke's model, we may consider this path as a completed structure).

Kripke presents us with a fascinating theorem concerning this model: if we say that ♀ forces *Q* whenever there exists a π in ♀ which forces *Q*, then for all *Q*, ♀ forces either *Q* or ~*Q*, *if and only if* ♀ *is generic*. This theorem elegantly brings together the essential law of classical logic—the point at which its difference from intuitionistic logic is concentrated—and the classical, non-intuitionistic concept *par excellence*: the actual completion of a extensionally determined, intrinsically non-constructible, infinite set.

Now, we must recall that throughout the entirety of ♀(*n*)'s historical existence, *n*, which marks both the 'age' of ♀(*n*) and its cardinality, remains finite. So long as ♀(*n*) is finite, it is *not yet a generic sequence*; it is merely *potentially* generic, but extensionally considered, it is no different than any other finite set in this respect. The Law of the Excluded Middle is therefore *not* generally valid for the subject of a truth procedure, insofar as this subject remains finite. The logic of the subject is not classical. It is intuitionistic.

§ 12

It is now possible to characterize the intuitionist application 'onto ontology [*i.e.* mathematics] rules of connection which *come from elsewhere*' (BE 249) in terms more precise and more rigorously developed than the vague epithets of 'empiricism' and 'objectivism' with which Badiou dismisses intuitionistic logic in the 24th Meditation. We may now characterize the logic of the intuitionist subject in terms internal to the conceptual apparatus set out in *Being and Event*: the rules of intuitionistic logic are precisely those prescribed by the law of the subject, the logic internal to a truth procedure. If intuitionist mathematics is justified in applying these rules back onto mathematics, it is because intuitionism seizes mathematics *as a truth procedure*. Conversely, if mathematics *is* a truth procedure, then these rules cannot be said to be derived from elsewhere; they are proceed from the very subjectivity which bears ontology towards truth.

The paradox, here, is that throughout *Being and Event*, mathematics is charged with a double task. It is repeatedly summoned not only to provide the ontological lineaments of the world, but also to stand as an exemplary truth procedure—indeed, as the paradigm for an entire species of truth procedures (the scientific). Yet if mathematics is a historical and concrete truth procedure, then its logic is not classical. And if mathematics is ontology, then either its logic cannot be the intuitionistic logic prescribed by the law of the subject, or else this ontology cannot be primarily set-theoretical.

Let us tackle one problem at a time: how is it possible for the logic of the mathematical truth procedure to be classical, when its subjective law is intuitionistic? In truth, the problem does not confront Badiou in this form, for he does not make of classical deduction the *law of the ontologist subject*. Instead, deduction is conceived as ontology's *operator of fidelity*, the principle whereby the ontologist subject concatenates the elements of the truth to which it is faithful (BE med. 24). Both classical and intuitionist deduction are conceived in this way, as two bifurcating regimes of fidelity (BE 249). Is this interpretation of deduction legitimate? Under the hypothesis that a truth is generic, it would seem that it is not. A sequence of elements concatenated in such a way that each is *deducible* from the series prior to it will not become generic. Insofar as classical mathematics is held to express, in its axioms, the laws of being *qua* being, the laws which dictate the formal structure *of any presentation whatsoever*, these laws are necessarily operative in the 'ontological situation' wherein the mathematician exercises her fidelity. Any sequence there articulated in accordance with these laws would be a *discernible* or *lawlike* sequence, and hence non-generic. Of course, the same problem would confront us if we chose to select *intuitionistic deducibility* as a principle of connection, but this is not the issue here. Deduction *as such* cannot be the principle of connection for a generic procedure. The principle of connection for mathematical truth procedures thus remains obscure. Of course, this is consonant with the nature of generic sequences: by definition, the operator of fidelity cannot be lawlike. It remains an open question how the operator of mathematical fidelity is to be thought. As for deduction, it can more consistently be conceived as the subjective law corresponding to the mathematical truth procedure, that is, as a manifestation of the forcing relation. And yet, if this is done, then deduction would obey an intuitionistic logic, and ontology, if it is a truth procedure, would not be classical.

If we maintain, despite all difficulties, that mathematics is a truth procedure in the sense outlined in *Being and Event*, the next question that we face concerns its status as ontology. This is a question that is far more profound and difficult that can be adequately dealt with here. A few, tentative remarks may be made at this point, however.

First of all, if the foregoing speculations are correct, then if mathematics is at once a truth and an ontology, then it would be compelled to obey an intuitionistic logic. This is not to say that it must be intuitionistic mathematics as such—as has been mentioned already, it would be wrong to reduce intuitionism to its abstract logical form. Nevertheless, this is a seemingly viable hypothesis. If we do take intuitionistic mathematics to be that which expresses the sayable of being, however, then we face the immediate consequence of having undercut a great deal of the formal apparatus that has brought us to this point. We lose the concept of the completed generic—even if such a figure never arrives historically, and we lose the non-wellfounded multiplicity that Badiou calls the event—even if such a structure was already foreclosed from the classical ontology. Time, on the other hand, enters into a much more subtle and organic relation with intuitionist ontology than it does with its classical predecessor, for which it appears as a somewhat awkward supplement. The question also arises as to the character of an intuitionistic ontology. There is no need to assume in advance that it would compel an idealist metaontology, as opposed to the materialist doctrine that Badiou sought to draw out of classical set theory; Badiou himself, at least in principle, wishes to distinguish between being in itself and what is *sayable of being*.[88] It is possible to uphold this distinction by maintaining beyond the scope of constructive thought, an unconstructed horizon about which we can, as of yet, say nothing.

Lest we loose the thread we took up at the beginning of this essay, let us take stock of the following points: Within the immanence of their procedures, the intuitionist and the post-evental subject are indiscernible from one another. It is their positions which differ. The post-evental subject is conceptually distinct from the intuitionistic subject in that its form is articulated within a medium that it did not create, and in that it proceeds from an aleatory event that is not the root of the ontological apparatus that delivers this medium (like the Brouwerian 'two-ity'), but an exception—something less than a sapling—that remains unthought by this very apparatus. Yet the ontological apparatus is *itself* to be conceived as a subjective procedure, and so we are driven to think the form of the Badiousian truth-subject within the field deployed by another subject of truth. If the subject of ontology is to formally coincide with the ontological schema of the subject, then we are presented with a problem, for the subject schematized by ontology is *incongruent* with the subjective form of ontology as such, insofar as this ontology is classical. If we insist on congruence, we are led away from classical ontology and towards intuitionism, but to take this route would require reformulating the problem to which we are responding.

At this point, the range of possible speculative solutions to these difficulties appears as broad as it is unclear. It seems that it would be both more fruitful and more cautious to formulate the general questions that confront us here. There are two:

88. cf. BE, p. 8: 'The thesis that I support does not in any way declare that being is mathematical, which is to say comprised of mathematical objectivities. It is not a thesis about the world but about discourse. It affirms that mathematics, throughout the entirety of its historical becoming, pronounces what is expressible of being qua being'.

What would it mean for ontology to be a truth procedure?

What would it mean for this not to be the case?

It remains to be seen whether they can be answered within the context in which they are posed.[89]

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89. With respect to Badiou's thought, I have intentionally restricted the focus of this paper to the system put forward in *Being and Event* (*L'être et l'événement*), and have not taken any consideration of the developments that this system has undergone in Badiou's 2006 work, *Logiques des mondes: l'être et l'événement 2*. This is a significant omission, given that the concept of the subject undergoes extensive revision in this recent work. Among the changes bearing on the above enquiry are a reworking of the subject in such a way that it is no longer simply the finite fragment of a truth, but participates in a properly infinite system of operations, as well as an explicit employment of the Heyting algebra for intuitionistic logic in the context of a theory of appearances that draws its mathematical support from category theory. A continuation of the current line of investigation into the terrain covered by *Logiques des mondes* is certainly called for, but this must await another time.

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